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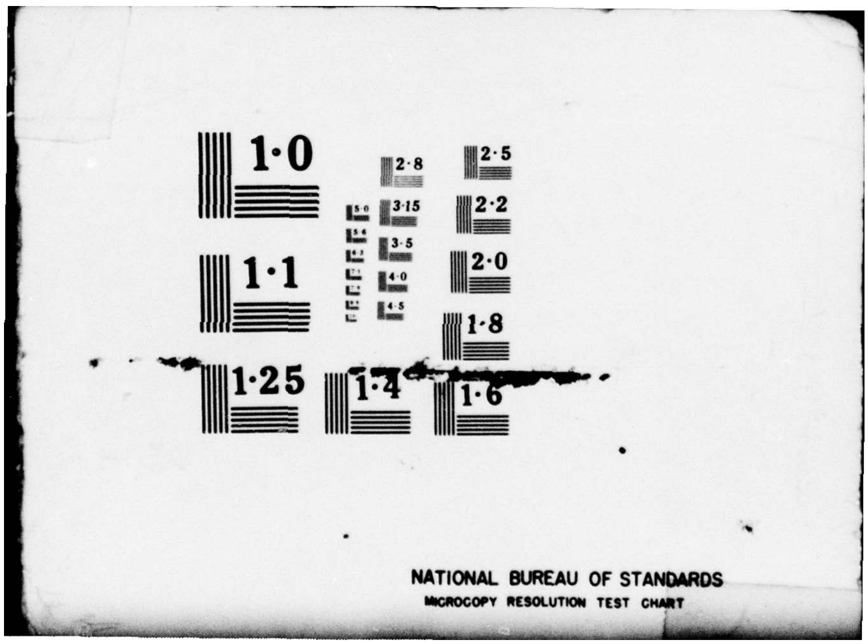
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RESEARCH REPORT 79-2

METHOD OF MOMENTS AS THE LEAST SQUARES SOLUTION FOR FITTING A POLYNOMIAL

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JUNE, 1979

Prepared under the contract number NOOO14-77-C-360, NR 150-402 with the Personnel and Training Research Programs Psychological Sciences Division Office of Naval Research

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(9) Research rept.,

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Research Report 79-2	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) Method of Moments as the Least Squares Solution for Fitting a Polynomial.		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Dr. Fumiko Samejima, Philip Livingston		8. CONTRACT OR GRANT NUMBER(s) (15) N00014-77-C-0360
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Psychology University of Tennessee Knoxville, TN 37916		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE: 61153N; PROJ: RR 042-04 TA: RR 042-04-01 WU: NR 150 -402
11. CONTROLLING OFFICE NAME AND ADDRESS Personnel and Training Research Programs Office of Naval Research (Code 458) Arlington, VA 22217		12. REPORT DATE (11) 16 June 79
		13. NUMBER OF PAGES 135 (12) 40p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (14) RR-79-2		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. Reproduction in whole or in part is permitted for any purpose of the United States government.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) (16) RR04204 (17) RR0420401		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Operating Characteristic Estimation Tailored Testing Latent Trait Theory		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (Please see reverse side) 404 225 JOB		

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Method of moments has been used frequently and effectively in our previous research for developing theories and methods of estimating the operating characteristics of the item response categories and ability distributions. It has been discovered that the method is quite useful, for fitting some curves to the set of observations, like maximum likelihood estimates, to the unobserved, conditional density function of which only the first few moments are estimated, and to the resultant, estimated density function of ability. It has also been discovered that polynomials are useful as functions to fit in applying the method of moments, with their unrestricted nature, regardless of the fact that there always is a possibility that they produce negative values for the estimated density. In the present paper, it is pointed out that a polynomial fitted by the method of moments is the same polynomial produced by the least squares principle. Using some examples, the two processes, i.e., the method of moments and the least squares solution, are compared. It is pointed out that, in general, the method of moments provides us with a simpler process and a more accurate result, than the least squares method, when the computer work is involved. The importance of using the appropriate interval in applying the method of moments, and the least squares method, is emphasized. In some examples, the results are compared with those by the polynomials obtained by Taylor's series.

METHOD OF MOMENTS AS THE LEAST SQUARES SOLUTION FOR FITTING A POLYNOMIAL

ABSTRACT

Method of moments has been used frequently and effectively in our previous research for developing theories and methods of estimating the operating characteristics of the item response categories and ability distributions. It has been discovered that the method is quite useful, for fitting some curves to the set of observations, like maximum likelihood estimates, to the unobserved, conditional density function of which only the first few moments are estimated, and to the resultant, estimated density function of ability. It has also been discovered that polynomials are useful as functions to fit in applying the method of moments, with their unrestricted nature, regardless of the fact that there always is a possibility that they produce negative values for the estimated density. In the present paper, it is pointed out that a polynomial fitted by the method of moments is the same polynomial produced by the least squares principle. Using some examples, the two processes, i.e., the method of moments and the least squares solution, are compared. It is pointed out that, in general, the method of moments provides us with a simpler process and a more accurate result, than the least squares method, when the computer work is involved. The importance of using the appropriate interval in applying the method of moments, and the least squares method, is emphasized. In some examples, the results are compared with those by the polynomials obtained by Taylor's series.

The research was conducted at the principal investigator's laboratory, 409 Austin Peay Hall, Department of Psychology, University of Tennessee, Knoxville, Tennessee. Those who worked in the laboratory and helped the authors in various ways for this research include Paul S. Changas, Lin Wen Kuei-Chiu, and Robert L. Trestman.

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I <u>Introduction</u>

Estimation of the operating characteristics of item response categories has been pursued and investigated by developing various methods and approaches (Samejima, 1977a, 1977b, 1978a, 1978b, 1978c, 1978d, 1978e, 1978f). Throughout these studies, the following two features are common.

(1) No prior mathematical formulae are assumed for the operating characteristics.

(2) A relatively small number of subjects are used to provide us with the set of data, on which the whole process of estimation is based.

Of these two features, the first one may be of more theoretical importance, since this will allow us to approach the genuine nature of the specific operating characteristic without preassuming it.

Another important feature of the studies of the operating characteristic estimation is that we have used the method of moments (Elderton and Johnson, 1969; Johnson and Kotz, 1970a, 1970b) for various purposes. To give several examples, we have used the method of moments for fitting a Pearson Type density function or a polynomial to the set of five hundred maximum likelihood estimates of ability, for fitting a Pearson Type density function to each conditional density of ability, given its maximum likelihood estimate, for fitting a polynomial for the set of calibrated ability scores $\tilde{\theta}$ for each binary item score group, and so on.

It is worth noting how well the method of moments fits into the whole process of estimation, and how important a role it has played. Above all, the method of moments for fitting a polynomial seems to be especially promising, since the polynomial is less restrictive in its shape even with a relatively low degree like degree 3 or 4, compared with other functions which require the same number of moments. One disadvantage of the polynomial is, however, that it is not a density function, and it may produce negative values for the estimated density function, or frequency function. Regardless of this possibility, we have used polynomials fitted by the method of moments for the estimated density functions frequently, and they have turned out to be quite successful.

In the present paper, the reason for this success is to be clarified. Also we are to expand the usefulness of the method of moments for fitting a polynomial beyond the scope of estimating a density function or a frequency function. Thus the method of moments is no longer just a method for fitting a curve to a frequency distribution, or to a set of observations, but it has become a useful way of providing us with a polynomial which approximates any observed and often non-mathematical function.

Some practical considerations are given in later sections of this paper, through several examples which are presented for the purpose of illustration.

II <u>Relationship between the Method of Moments for Fitting a Polynomial and the Least Squares Principle</u>

Let $h(t)$ be any function of the variable t, which is defined in a closed interval, $[\underline{t}, \bar{t}]$, and is integrable in the Lebesgue sense and has the first m moments. This function $h(t)$ can be some specified mathematical function, or an empirically obtained function. Let α_i $(i = 0, 1, \ldots, m)$ be the i-th coefficient of the polynomial which can be written in the form

$$(2.1) \qquad \sum_{i=0}^{m} \alpha_i t^i ,$$

and is to be fitted to the function $h(t)$ following the least squares principle. We define Q such that

$$(2.2) \qquad 2Q = \int_{\underline{t}}^{\bar{t}} [h(t) - \sum_{i=0}^{m} \alpha_i t^i]^2 \, dt .$$

Differentiating Q with respect to α_r and setting the result equal to zero, we obtain

$$(2.3) \qquad \frac{\partial Q}{\partial \alpha_r} = \int_{\underline{t}}^{\bar{t}} [h(t) - \sum_{i=0}^{m} \alpha_i t^i][-t^r] \, dt = 0$$

and then

$$(2.4) \qquad \int_{\underline{t}}^{\bar{t}} t^r h(t) \, dt = \int_{\underline{t}}^{\bar{t}} t^r \sum_{i=0}^{m} \alpha_i t^i \, dt ,$$

for $r = 0, 1, 2, \ldots, m$.

Thus it is obvious from (2.3) that the least squares principle requires the resultant polynomial of degree m to have the same 0-th through m-th moments as $h(t)$, which is nothing but the principle upon which the method of moments is based. From this result, it is obvious that both methods provide us with the same polynomial.

It is clear, therefore, that the method of moments can be used as a substitute for the least squares method when we fit a polynomial of any degree to a function, which may or may not assume negative values as well.

When the function $h(t)$ is observed only at N points of the variable t, as is often the case for an empirically observed function, we can replace (2.2) by

$$2Q = \sum_{k=1}^{N} \{[h(t_k) - \sum_{i=0}^{m} \alpha_i t_k^{\ i}]\, w(t_k)\}^2 , \tag{2.5}$$

where $w(t_k)$ is some appropriately chosen weight for t_k. Differentiating (2.5) and setting the result equal to zero, we obtain

$$\sum_{k=1}^{N} t_k^{\ r} h(t_k)\, w(t_k) = \sum_{k=1}^{N} t_k^{\ r} w(t_k) \sum_{i=0}^{m} \alpha_i t_k^{\ i} . \tag{2.6}$$

If the function $h(t)$ is continuous and we divide the interval $[\underline{t}, \bar{t}]$ into N subintervals, by the middle value theorem there exists, at least, one value, ζ_{kr}, in each subinterval $(\underline{t}_k, \bar{t}_k)$ which satisfies

$$\int_{\underline{t}_k}^{\bar{t}_k} t^r h(t)\, dt = \zeta_{kr}^r\, h(\zeta_{kr})(\bar{t}_k - \underline{t}_k) , \tag{2.7}$$

where

$$\bar{t}_k = \underline{t}_{k+1} \tag{2.8}$$

for $k = 1, 2, \ldots, (N-1)$, and

$$\begin{cases} \underline{t}_1 = \underline{t} \\ \bar{t}_N = \bar{t} \end{cases} . \tag{2.9}$$

When the width of each subinterval is small enough, these $(m+1)$ values, ζ_{kr} $(r = 0, 1, 2, \ldots, m)$, can be approximated by a single value, say,

the midpoint of the subinterval. Using such a value as t_k and the subinterval width as $w(t_k)$, we can approximate (2.2) by (2.6). If all the subinterval widths are equal, (2.6) is simplified to provide

$$(2.10) \qquad \sum_{k=1}^{N} t_k{}^r \, h(t_k) = \sum_{k=1}^{N} t_k{}^r \sum_{i=0}^{m} \alpha_i \, t_k{}^i \, .$$

III Least Squares Solution

We can rewrite (2.4) in the form

$$\mu'_s = \sum_{j=1}^{m+1} \alpha_{j-1}\,[j+s-1]^{-1}\,[\bar{t}^{\,j+s-1} - \underline{t}^{\,j+s-1}] \,, \tag{3.1}$$

where $s = r + 1 = 1, 2, \ldots, m+1$, $j = i + 1 = 1, 2, \ldots, m+1$, and μ'_s is the (s-1)-th moment of t about the origin, defined by

$$\mu'_s = \int_{\underline{t}}^{\bar{t}} t^{s-1}\, h(t)\, dt \,. \tag{3.2}$$

Let α be a column vector of order (m+1), whose j-th element is α_{j-1}, and μ' be a column vector of the same order whose s-th element is μ'_s. Thus we can rewrite (3.1) in the matrix notation to obtain

$$\mu' = A\alpha \,, \tag{3.3}$$

where A is a symmetric matrix of order (m+1) whose sj-element is given by

$$[j+s-1]^{-1}\,[\bar{t}^{\,j+s-1} - \underline{t}^{\,j+s-1}] \,. \tag{3.4}$$

The least squares solution for α is obtained, therefore, by

$$\hat{\alpha} = A^{-1}\mu' \,. \tag{3.5}$$

For the purpose of illustration, the matrix A for $m = 2$ is shown below as an example.

$$A = \begin{bmatrix} (\bar{t} - \underline{t}) & (\bar{t}^2 - \underline{t}^2)/2 & (\bar{t}^3 - \underline{t}^3)/3 \\ (\bar{t}^2 - \underline{t}^2)/2 & (\bar{t}^3 - \underline{t}^3)/3 & (\bar{t}^4 - \underline{t}^4)/4 \\ (\bar{t}^3 - \underline{t}^3)/3 & (\bar{t}^4 - \underline{t}^4)/4 & (\bar{t}^5 - \underline{t}^5)/5 \end{bmatrix} \,. \tag{3.6}$$

In practice, we usually use a greater value for m, and obtaining the inverse matrix of A will be the most intricate process of computation

and the availability of a package program for inversing a symmetric matrix will be of necessity.

IV Solution by the Method of Moments

Let $R(t)$ be a half of the interval width for which the function $h(t)$ is defined, and $M(t)$ be the midpoint of the interval, such that

$$R(t) = (\bar{t} - \underline{t})/2 \tag{4.1}$$

and

$$M(t) = (\bar{t} + \underline{t})/2 \ . \tag{4.2}$$

For convenience, we define a new variable t^* by changing the origin of t to the midpoint of the interval $[\underline{t}, \bar{t}]$, i.e.,

$$t^* = t - M(t) \ . \tag{4.3}$$

Thus the polynomial of degree m in t can be rewritten as a polynomial of the same degree in t^*, or

$$\sum_{i=0}^{m} \alpha_i t^i = \sum_{i=0}^{m} a_i t^{*i} \ , \tag{4.4}$$

with the relationship between the two sets of coefficients such that

$$\alpha_r \begin{cases} = a_r & \text{for } M(t) = 0 \\ = \sum_{i=r}^{m} (-1)^{i-r} a_i \binom{i}{r} [M(t)]^{i-r} \ , & \text{otherwise,} \end{cases} \qquad r = 0,1,\ldots,m \ . \tag{4.5}$$

To give an example, when $m = 7$, we can rewrite (4.5) as follows.

$$\begin{cases} \alpha_0 = a_0 - a_1 M(t) + a_2[M(t)]^2 - a_3[M(t)]^3 + a_4[M(t)]^4 \\ \qquad - a_5[M(t)]^5 + a_6[M(t)]^6 - a_7[M(t)]^7 \\ \alpha_1 = a_1 - 2a_2 M(t) + 3a_3[M(t)]^2 - 4a_4[M(t)]^3 + 5a_5[M(t)]^4 \\ \qquad - 6a_6[M(t)]^5 + 7a_7[M(t)]^6 \\ \alpha_2 = a_2 - 3a_3 M(t) + 6a_4[M(t)]^2 - 10a_5[M(t)]^3 + 15a_6[M(t)]^4 \\ \qquad - 21a_7[M(t)]^5 \\ \alpha_3 = a_3 - 4a_4 M(t) + 10a_5[M(t)]^2 - 20a_6[M(t)]^3 + 35a_7[M(t)]^4 \end{cases} \tag{4.6}$$

$$\begin{cases} \alpha_4 = a_4 - 5a_5M(t) + 15a_6[M(t)]^2 - 35a_7[M(t)]^3 \\ \alpha_5 = a_5 - 6a_6M(t) + 21a_7[M(t)]^2 \\ \alpha_6 = a_6 - 7a_7M(t) \\ \alpha_7 = a_7 \ . \end{cases}$$

The following relationships hold between the moments about the midpoint $M(t)$ and the coefficients a_r $(r = 1, 2, \ldots, m)$.*

$$\mu^*_{2g} = 2 \sum_{k=0}^{[m/2]} a_{2k} [2(g+k)+1]^{-1} [R(t)]^{2(g+k)+1}, \quad g = 0,1,2,\ldots,[m/2] \tag{4.7}$$

$$\mu^*_{2g+1} = 2 \sum_{k=0}^{[(m-1)/2]} a_{2k+1} [2(g+k+1)+1]^{-1} [R(t)]^{2(g+k+1)+1}, \quad g = 0,1,2,\ldots,[(m-1)/2] \tag{4.8}$$

In the above two equations, [ ] indicates the integer part of the number, and μ^*_{2g} and μ^*_{2g+1} indicate even and odd moments about the midpoint $M(t)$ respectively.

Let $p = g + 1$ and $q = k + 1$. We define the following two symmetric matrices, $B_{(0)}$ and $B_{(1)}$, whose orders are both $(m+1)/2$ when m is odd, and $(m/2)+1$ and $(m/2)$ when m is even, respectively.

$$B_{(0)} = \{ [R(t)]^{2(p+q)-3} [2(p+q)-3]^{-1} \} . \tag{4.9}$$

$$B_{(1)} = \{ [R(t)]^{2(p+q)-1} [2(p+q)-1]^{-1} \} . \tag{4.10}$$

Let $\mu^*_{(0)}$ and $\mu^*_{(1)}$ be column vectors of the corresponding orders, such that

$$\mu^*_{(0)} = \{ \mu^*_{2(p-1)} \}' , \qquad p = 1,2,\ldots,[m/2]+1 \tag{4.11}$$

and

$$\mu^*_{(1)} = \{ \mu^*_{2p-1} \}' . \qquad p = 1,2,\ldots,[(m+1)/2] \tag{4.12}$$

* For further details, cf. Elderton and Johnson, 1969, or Appendix I.

Let $a_{(0)}$ and $a_{(1)}$ denote the coefficient vectors of the corresponding orders, which can be written as

(4.13) $$a_{(0)} = \{ a_{2(q-1)} \}' \ , \qquad q = 1,2,\ldots,[m/2]+1 \ ,$$

and

(4.14) $$a_{(1)} = \{ a_{2q-1} \}' \ , \qquad q = 1,2,\ldots,[(m+1)/2] \ .$$

Thus we can rewrite (4.7) and (4.8) in the matrix notation such that

(4.15) $$\mu^*_{(0)} = 2\, B_{(0)}\, a_{(0)}$$

and

(4.16) $$\mu^*_{(1)} = 2\, B_{(1)}\, a_{(1)} \ .$$

The coefficient matrices $a_{(0)}$ and $a_{(1)}$ are obtained, therefore, by

(4.17) $$a_{(0)} = (1/2)\, B^{-1}_{(0)}\, \mu^*_{(0)}$$

and

(4.18) $$a_{(1)} = (1/2)\, B^{-1}_{(1)}\, \mu^*_{(1)} \ .$$

In practice, the computation is facilitated if we define two matrices, $C_{(0)}$ and $C_{(1)}$, of orders $[m/2]+1$ and $[(m+1)/2]$, respectively, such that

(4.19) $$C_{(0)} = \{ [2(p+q)-3]^{-1} \}$$

and

(4.20) $$C_{(1)} = \{ [2(p+q)-1]^{-1} \} \ ,$$

which do not depend on a specific set of data but depend only upon the degree of the polynomial. From these two matrices, we can obtain the

two matrices, $(1/2)\ C_{(0)}^{-1}$ and $(1/2)\ C_{(1)}^{-1}$, and it is easily seen that $(1/2)\ B_{(0)}^{-1}$ and $(1/2)\ B_{(1)}^{-1}$ are obtained by dividing the element in the p-th row and q-th column of the corresponding matrices by $[R(t)]^{2(p+q)-3}$ and $[R(t)]^{2(p+q)-1}$, respectively, for every combination of p and q. For the purpose of *illustration*, the process of obtaining $(1/2)\ C_{(0)}^{-1}$ and $(1/2)\ C_{(1)}^{-1}$ is shown for $m = 6$ and $m = 7$ in Appendix II. The resultant sets of equations for obtaining the coefficients a_i are listed below for the polynomials of degrees 3, 4, 5, 6 and 7 .

(i) Polynomial of Degree 3

$$
(4.21)\quad \left\{
\begin{aligned}
a_0 &= [1.125\mu_0^*/R] - [1.875\mu_2^*/R^3] \\
a_1 &= [9.375\mu_1^*/R^3] - [13.125\mu_3^*/R^5] \\
a_2 &= [-1.875\mu_0^*/R^3] + [5.625\mu_2^*/R^5] \\
a_3 &= [-13.125\mu_1^*/R^5] + [21.875\mu_3^*/R^7]
\end{aligned}
\right.
$$

(ii) Polynomial of Degree 4

$$
(4.22)\quad \left\{
\begin{aligned}
a_0 &= [1.7578125\mu_0^*/R] - [8.203125\mu_2^*/R^3] + [7.3828125\mu_4^*/R^5] \\
a_1 &= [9.375\mu_1^*/R^3] - [13.125\mu_3^*/R^5] \\
a_2 &= [-8.203125\mu_0^*/R^3] + [68.90625\mu_2^*/R^5] - [73.828125\mu_4^*/R^7] \\
a_3 &= [-13.125\mu_1^*/R^5] + [21.875\mu_3^*/R^7] \\
a_4 &= [7.3828125\mu_0^*/R^5] - [73.828125\mu_2^*/R^7] + [86.1328125\mu_4^*/R^9]
\end{aligned}
\right.
$$

(iii) Polynomial of Degree 5

$$
(4.23)\quad \left\{
\begin{aligned}
a_0 &= [1.7578125\mu_0^*/R] - [8.203125\mu_2^*/R^3] + [7.3828125\mu_4^*/R^5] \\
a_1 &= [28.7109375\mu_1^*/R^3] - [103.359375\mu_3^*/R^5] + [81.2109375\mu_5^*/R^7] \\
a_2 &= [-8.203125\mu_0^*/R^3] + [68.90625\mu_2^*/R^5] - [73.828125\mu_4^*/R^7] \\
a_3 &= [-103.359375\mu_1^*/R^5] + [442.96875\mu_3^*/R^7] - [378.984375\mu_5^*/R^9] \\
a_4 &= [7.3828125\mu_0^*/R^5] - [73.828125\mu_2^*/R^7] + [86.1328125\mu_4^*/R^9] \\
a_5 &= [81.2109375\mu_1^*/R^7] - [378.984375\mu_3^*/R^9] + [341.0859375\mu_5^*/R^{11}]
\end{aligned}
\right.
$$

(iv) Polynomial of Degree 6

$$
(4.24)\qquad
\left\{
\begin{aligned}
a_0 &\doteq [2.3925781\mu_0^*/R] - [21.5332031\mu_2^*/R^3] + [47.3730469\mu_4^*/R^5] \\
&\qquad - [29.3261719\mu_6^*/R^7] \\
a_1 &= [28.7109375\mu_1^*/R^3] - [103.359375\mu_3^*/R^5] + [81.2109375\mu_5^*/R^7] \\
a_2 &\doteq [-21.5332031\mu_0^*/R^3] + [348.8378906\ {}_2^*/R^5] \\
&\qquad - [913.6230469\mu_4^*/R^7] + [615.8496094\mu_6^*/R^9] \\
a_3 &= [-103.359375\mu_1^*/R^5] + [442.96875\mu_3^*/R^7] - [378.984375\mu_5^*/R^9] \\
a_4 &\doteq [47.3730469\mu_0^*/R^5] - [913.6230469\mu_2^*/R^7] \\
&\qquad + [2605.5175781\mu_4^*/R^9] - [1847.5488281\mu_6^*/R^{11}] \\
a_5 &= [81.2109375\mu_1^*/R^7] - [378.984375\mu_3^*/R^9] + [341.0859375\mu_5^*/R^{11}] \\
a_6 &\doteq [-29.3261719\mu_0^*/R^7] + [615.8496094\mu_2^*/R^9] \\
&\qquad - [1847.5488281\mu_4^*/R^{11}] + [1354.8691406\mu_6^*/R^{13}]
\end{aligned}
\right.
$$

(v) Polynomial of Degree 7

$$
(4.25)\qquad
\left\{
\begin{aligned}
a_0 &\doteq [2.3925781\mu_0^*/R] - [21.5332031\mu_2^*/R^3] + [47.3730469\mu_4^*/R^5] \\
&\qquad - [29.3261719\mu_6^*/R^7] \\
a_1 &\doteq [64.5996094\mu_1^*/R^3] - [426.3574219\mu_3^*/R^5] \\
&\qquad + [791.8066406\mu_5^*/R^7] - [439.8925781\mu_7^*/R^9] \\
a_2 &\doteq [-21.5332031\mu_0^*/R^3] + [348.8378906\ {}_2^*/R^5] \\
&\qquad - [913.6230469\mu_4^*/R^7] + [615.8496094\mu_6^*/R^9] \\
a_3 &\doteq [-426.3574219\mu_1^*/R^5] + [3349.9511719\mu_3^*/R^7] \\
&\qquad - [6774.3457031\mu_5^*/R^9] + [3959.0332031\mu_7^*/R^{11}] \\
a_4 &\doteq [47.3730469\mu_0^*/R^5] - [913.6230469\mu_2^*/R^7] \\
&\qquad + [2605.5175781\mu_4^*/R^9] - [1847.5488281\mu_6^*/R^{11}] \\
a_5 &\doteq [791.8066406\mu_1^*/R^7] - [6774.3457031\mu_3^*/R^9] \\
&\qquad + [14410.8808594\mu_5^*/R^{11}] - [8709.8730469\mu_7^*/R^{13}] \\
a_6 &\doteq [-29.3261719\mu_0^*/R^7] + [615.8496094\mu_2^*/R^9] \\
&\qquad - [1847.5488281\mu_4^*/R^{11}] + [1354.8691406\mu_6^*/R^{13}] \\
a_7 &\doteq [-439.8925781\mu_1^*/R^9] + [3959.0332031\mu_3^*/R^{11}] \\
&\qquad - [8709.8730469\mu_5^*/R^{13}] + [5391.8261719\mu_7^*/R^{15}]
\end{aligned}
\right.
$$

(For simplicity, in the above equations, R is used instead of $R(t)$.)

A computer program specially written for this purpose can handle the above sets of equations for obtaining the coefficients a_i and then (4.5) for obtaining the revised coefficients α_i, for $i = 0, 1, \ldots, m$, with the $(m+1)$ moments about the midpoint, μ_k^*, the midpoint of the interval, $M(t)$, and a half of the interval width, $R(t)$, as the input data. As a substitute for the least squares solution, this will be much easier since it does not include an inversion process for the matrix A defined in the preceding section.

V Comparison of a Polynomial with a Pearson Type Density Function As an Estimated Density Function

It was observed in Section 2 that the polynomial obtained by the method of moments is the one which fits best to any function among all the polynomials of the same degree, from the standpoint of the least squares principle. This is true whether the function is empirically obtained, or mathematically defined, since $h(t)$ in Section 2 does not require any such restrictions. Considering the less restrictive nature of polynomials in comparison with density functions of fixed types, the above characteristic of the polynomial obtained by the method of moments makes us expect the one to provide us with a better fitting curve in most situations. For this reason, it will be meaningful to compare some of the results we have already produced using the method of moments.

The data used here are the set of five hundred maximum likelihood estimates used in the studies of the estimation of the operating characteristics of item response categories (Samejima, 1977a, 1977b, 1978a, 1978b, 1978c, 1978d, 1978e, 1978f). They are simulated data, based on five hundred hypothetical examinees, whose ability levels are located at one hundred equally distanced points between -2.475 and 2.475 inclusive on the ability scale, with the interval length of 0.05 and five subjects at each point. Their maximum likelihood estimates are obtained on their response patterns, which were calibrated by Monte Carlo Method, on the set of thirty-five graded response items, which is called Old Test, and whose test information function is approximately equal to 21.63 for the range of ability θ, -3.0 through 3.0. The funda-

mental concepts and formulae in these processes are given in Appendix III, where the maximum likelihood estimate $\hat{\theta}$ is replaced by the more general estimate, λ . The moments were computed for this set of five hundred observations.

The first moment about the origin, or the mean of the set of observations, turned out to be -0.00577 , which is very close to zero. The second through fifth moments about the mean are 2.14824 , -0.01465 , 8.65145 and -0.25790 . Figure 5-1 presents the polynomial of degree 4 obtained by the method of moments, together with the theoretical density function and the histogram of the frequency distribution of the five hundred observations categorized, for convenience, into twenty-five subintervals of the width of 0.25 . (Actually, in this graph and in those others shown later in the present section, the ordinate is the frequency instead of the density. Thus frequency 40, for example, corresponds to density 0.32, 30 to 0.24, and so forth.) We can see that the polynomial of degree 4 thus obtained shows a good fit to both the theoretical density and the empirical data.

In obtaining the above polynomial of degree 4, the first four moments of $\hat{\theta}$ are used. This same set of moments enables us to obtain Pearson's criterion κ (Elderton and Johnson, 1969, Johnson and Kotz, 1970a, 1970b) for the selection of the suitable Pearson Type distribution for our data. The formula for Pearson's criterion is such that

$$\kappa = \beta_1(\beta_2+3)^2[4(2\beta_2-3\beta_1-6)(4\beta_2-3\beta_1)]^{-1} \quad (5.1)$$

where β_1 and β_2 are defined in terms of the second through fourth moments about the mean, which are denoted by μ_2 , μ_3 and μ_4 ,

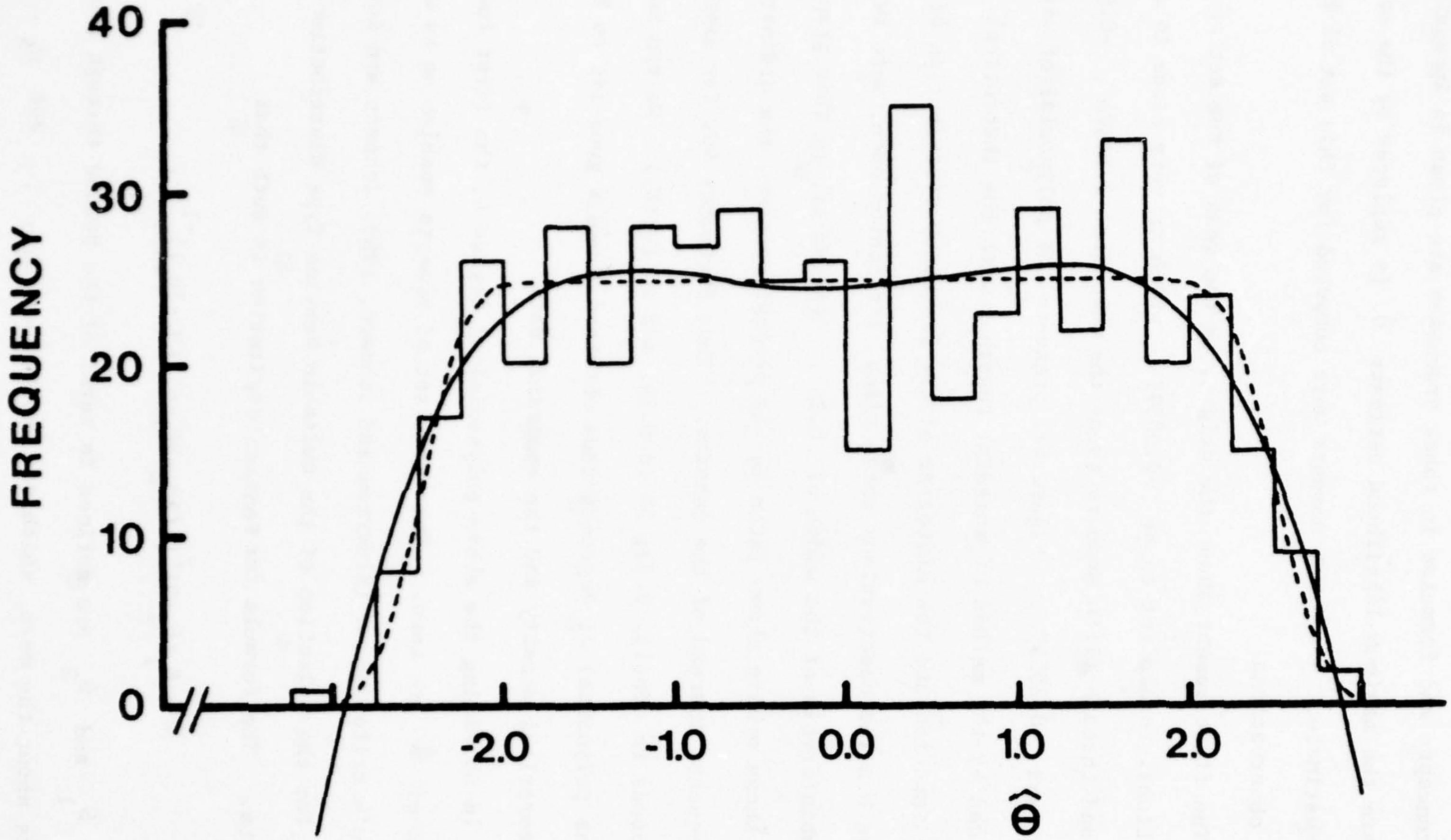

FIGURE 5-1

Frequency *Distribution of the Maximum Likelihood* Estimate (Histogram), Its Approximated Polynomial of Degree 4 (Solid Curve) and the Theoretical Density of the Maximum Likelihood Estimate (Dotted Curve).

respectively, by

$$\beta_1 = \mu_3^2 \, \mu_2^{-3} \tag{5.2}$$

and

$$\beta_2 = \mu_4 \, \mu_2^{-2} \quad . \tag{5.3}$$

The selection is made in such a way that, if, for instance, κ turned out to be finite and negative, then the distribution will be of Type I; if it turned out to be positive and less than unity, then the distribution will be of Type IV; and so on.

With our data, using the second, third and fourth moments about the mean, which were given earlier, the value of Pearson's criterion κ is -0.00000762 . This value is very close to zero, and Pearson's Type II distribution should be selected. This distribution is a special case of Beta distribution, or Pearson's Type I distribution, whose density function is given by

$$[B(p,q)]^{-1}(\hat{\theta}-a)^{p-1}(b-\hat{\theta})^{q-1}(b-a)^{-(p+q-1)}, \tag{5.4}$$

where $B(p,q)$ is the Beta function with parameters, p and q , and for the Type II distribution we have

$$q = p \quad . \tag{5.5}$$

The three parameters, a , b and p , of the Type II density function can be estimated

$$p = 3(\beta_2-1)[2(3-\beta_2)]^{-1} \quad , \tag{5.6}$$

$$(5.7) \qquad a = \mu_1' - (2\mu_4\mu_2^{-1})^{1/2}(3-\beta_2)^{-1/2}$$

and

$$(5.8) \qquad b = \mu_1' + (2\mu_4\mu_2^{-1})^{1/2}(3-\beta_2)^{-1/2} ,$$

respectively, where μ_1' is the mean of the maximum likelihood estimate and μ_2 is its variance. The estimated parameters thus obtained turned out to be 1.16581, -2.68111 and 2.66947, respectively. Figure 5-2 presents the resultant Pearson's Type II density function, together with the theoretical density and the histogram, which were also presented in Figure 5-1.

It should be noted that the theoretical density of $\hat{\theta}$, which is drawn in Figures 5-1 and 5-2, is symmetric. This was obtained by setting $f(\theta) = (\bar{\theta}-\underline{\theta})^{-1}$ with $\underline{\theta} = -2.5$ and $\bar{\theta} = 2.5$ for the marginal density of θ and $n(\theta,0.215)$ for $\psi(\hat{\theta}|\theta)$, the conditional density of $\hat{\theta}$, given θ (cf. Appendix III). The resultant theoretical density is given by

$$(5.9) \qquad g(\hat{\theta}) = 0.2(2\pi)^{-1/2} \int_{(-2.5-\hat{\theta})/0.215}^{(2.5-\hat{\theta})/0.215} \exp[-t^2/2]\, dt \quad .$$

We must say, therefore, that the selection of the Type II distribution out of all the Pearson Type distributions was a good one, providing us with a uni-modal curve for the estimated density function, as we can see in Figure 5-2. Regardless of this fact, however, we note that the fit of the polynomial of degree 4, which is drawn in Figure 5-1, to the theoretical density is still better than that of the Pearson's Type II density function. This result can be considered as evidence

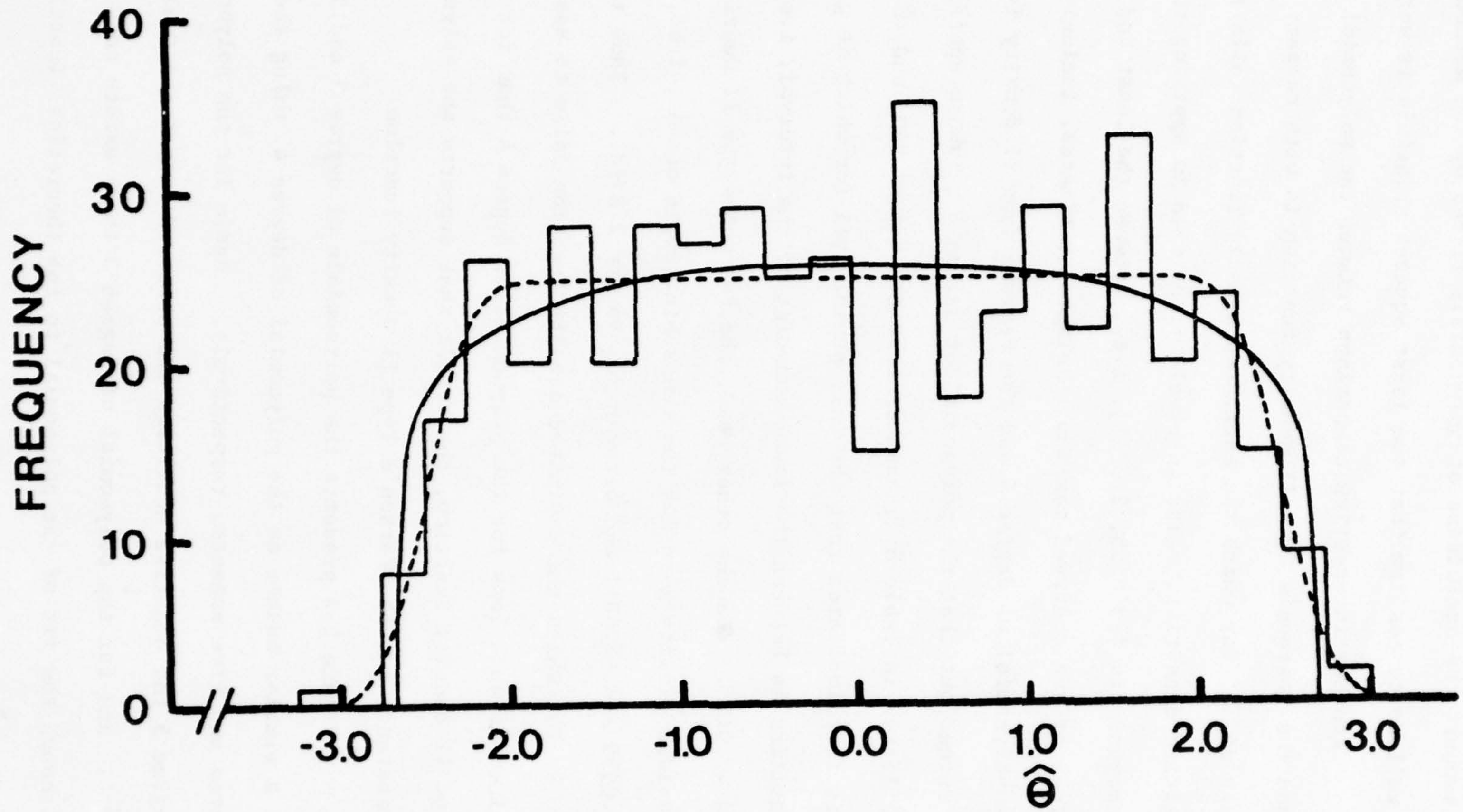

FIGURE 5-2

Frequency Distribution of the Maximum Likelihood Estimate (Histogram), Its Approximated Pearson's Type II Density Function (Solid Curve) and the Theoretical Density of the Maximum Likelihood Estimate (Dotted Curve).

to support the usefulness of polynomials fitted by the method of moments, as well as to confirm that the least squares principle is working well.

Another interesting comparison between the polynomial of degree 4 and the Pearson's Type II density function is with respect to the range of $\hat{\theta}$ for which the estimated density function fails to provide positive densities. The interval of $\hat{\theta}$ used in applying the method of moments is [-3.0555, 2.8718], i.e., between the least and the greatest values of the observed maximum likelihood estimates, inclusive, for both the polynomial of degree 4 and the Pearson Type II density function. It turned out that the polynomial of degree 4 , whose coefficients are listed in Table 5-1, assumes zero at $\hat{\theta} \doteq -2.940$ and $\hat{\theta} \doteq 2.891$. This fact indicates that the only subinterval for which it gives negative densities is between the lower endpoint of the interval, i.e., -3.0555 , and -2.940 . On the other hand, the Pearson's Type II density function provides density zero for the two subintervals of $\hat{\theta}$, i.e., between -3.0555 and -2.6811 and between 2.6695 and 2.8718 . Thus the range of $\hat{\theta}$ for which the estimated density function fails to assume positive values is much less for the polynomial of degree 4 than for the Pearson's Type II density function, the result that supports the polynomial in preference to the Pearson's Type II density function.

Figure 5-3 presents the polynomials of degree 3 and 5 obtained in a similar manner as the polynomial of degree 4, using the first three and five moments, respectively. Again for the polynomial of degree 5 the negative densities are observed only for a small range of $\hat{\theta}$, and for the polynomial of degree 3 there exists no such range, although the fit of the polynomial to the theoretical function is not

TABLE 5-1

Coefficients α_i's of the Resultant Polynomials of Degrees 3, 4, and 5

i	DGR. 3	DGR. 4	DGR. 5
0	0.22416	0.19620	0.19539
1	-0.00351	0.00238	-0.00638
2	-0.01873	0.01319	0.01449
3	0.00095	-0.00062	0.00405
4		-0.00427	-0.00449
5			-0.00048

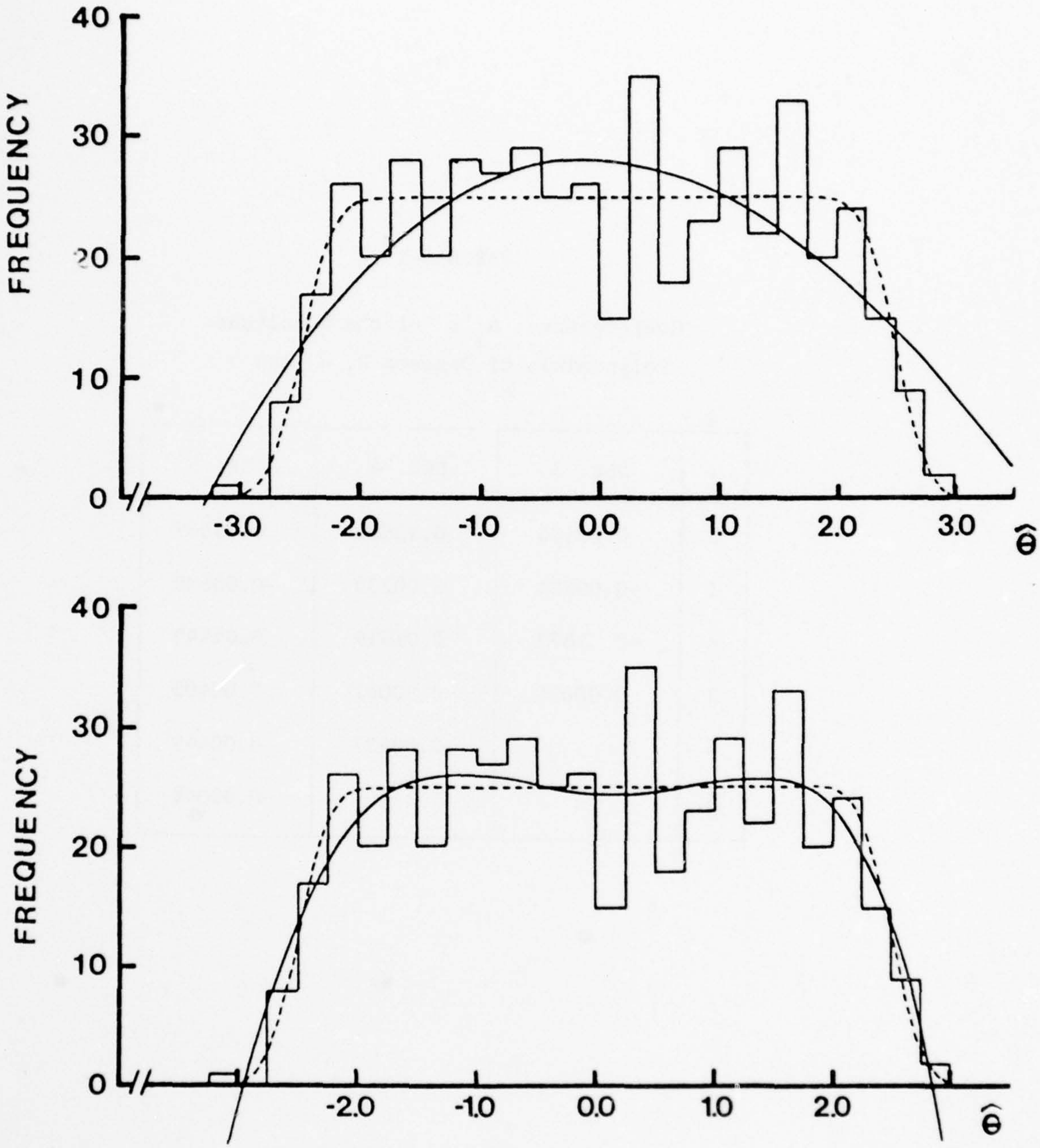

FIGURE 5-3

Frequency Distribution of the Maximum Likelihood Estimate (Histogram), Its Approximated Polynomial (Solid Curve) and the Theoretical Density of the Maximum Likelihood Estimate (Dotted Curve). The Degree of the Polynomial Is Three in the Upper Graph, and Five in the Lower Graph.

so good as in the case of the polynomial of degree 4 or 5 .

We notice that the polynomials of degrees 4 and 5 are very close to each other, and yet there is a slight tendency that the one with degree 5 is slightly more departing from the theoretical curve, than the one with degree 4. This is understandable considering the fact that the polynomial of degree 5 uses an additional moment, i.e., the fifth moment, and that the sampling fluctuations of moments of higher degrees are greater than those of lower degrees. This implies the warning that we should balance the sample size and the degree of the polynomial to be adopted in the method of moments, in order to make a right selection of the degree. Figures 5-4 and 5-5 are the reproduction of the results (Samejima, 1977b) obtained by fitting a polynomial to each subset of the five hundred observations, using the polynomials of degrees 3 and 4. In the results shown as Figure 5-4, each subset contains 250 observed maximum likelihood estimates, and in those shown as Figure 5-5 each has only 100 observations. We can see that, especially in the latter case, even the polynomials of degree 3 are influenced by the moments of this particular sample, and fail to provide a set of curves close to the the theoretical density function.

The coefficients of the polynomials fitted for the two subsets and those for the five subsets are presented as Tables 5-2 and 5-3, respectively.

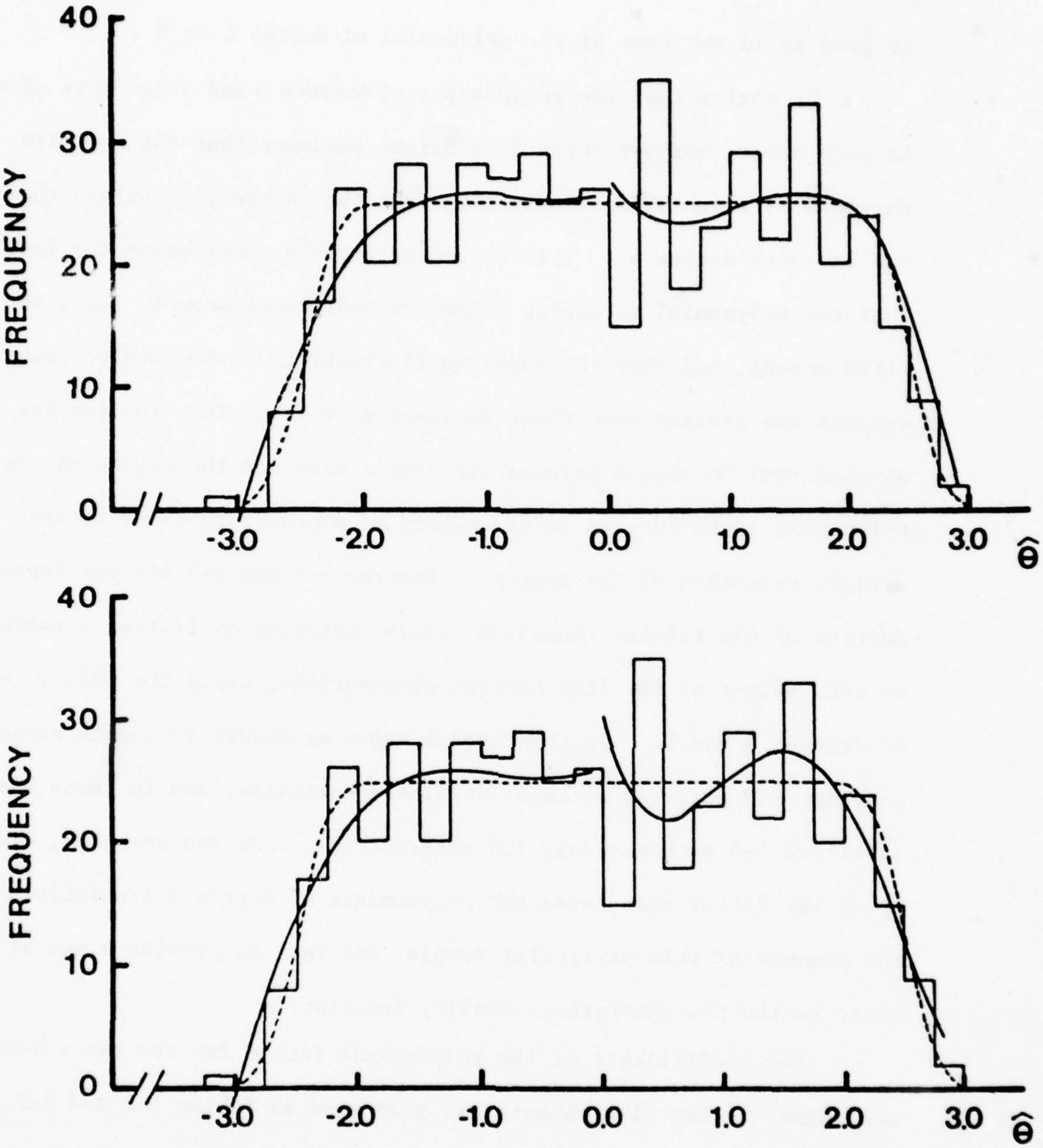

FIGURE 5-4

Two Polynomials Approximating the Two Subsets of Five Hundred Maximum Likelihood Estimates, Respectively (Solid Curve), the Frequency Distribution of the Maximum Likelihood Estimate (Histogram) and the Theoretical Density of the Maximum Likelihood Estimate (Dotted Curve). The Degrees of the Two Polynomials Are Three in the Upper Graph, and Four in the Lower Graph.

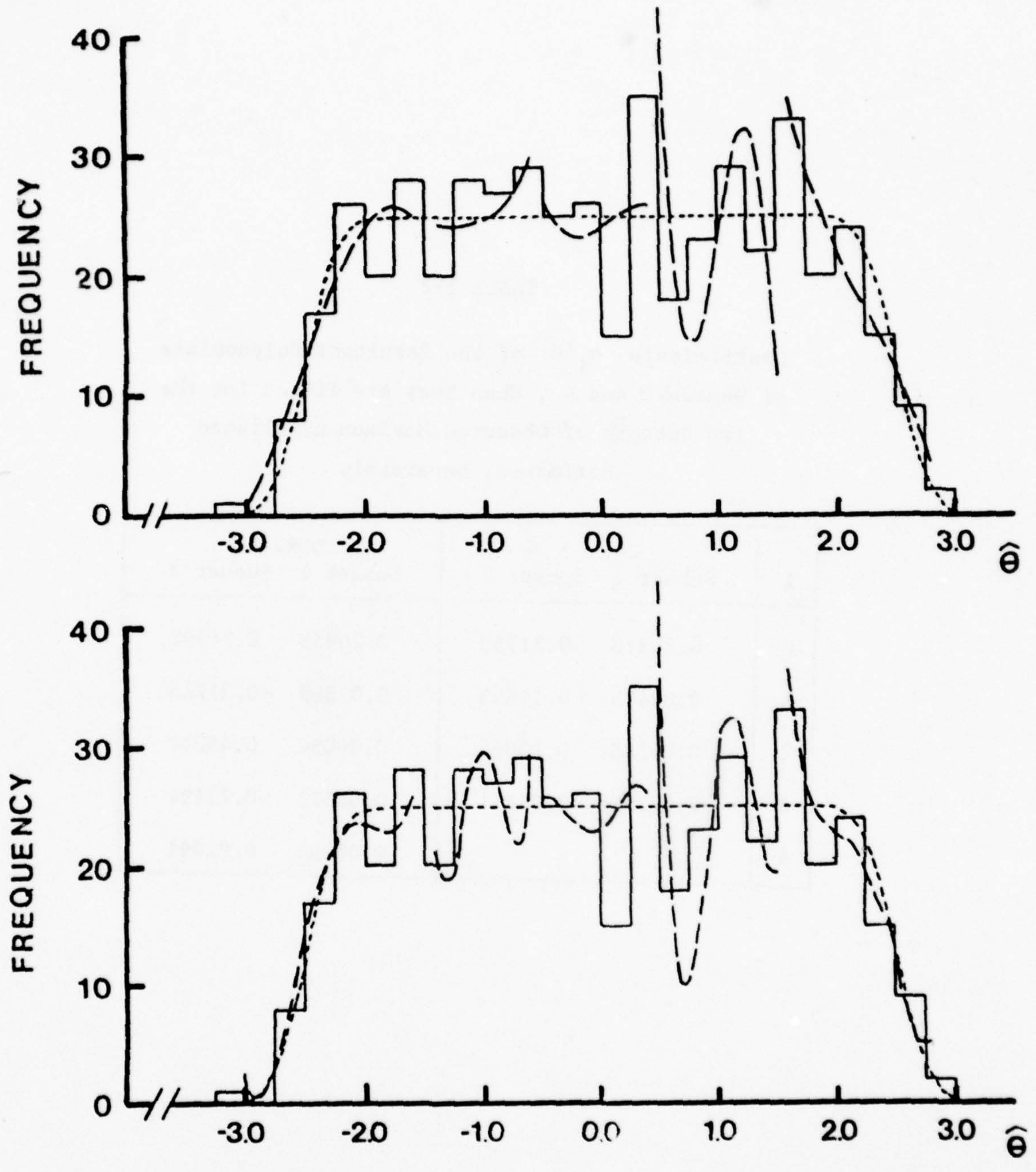

FIGURE 5-5

Five Polynomials Approximating the Five Subsets of Five Hundred Maximum Likelihood Estimates, Respectively (Dashed Curve), the Frequency Distribution of the Maximum Likelihood Estimate (Histogram) and the Theoretical Density of the Maximum Likelihood Estimate (Dotted Curve). The Degrees of the Two Polynomials Are Three in the Upper Graph, and Four in the Lower Graph.

TABLE 5-2

Coefficients α_i's of the Resultant Polynomials of Degrees 3 and 4, When They Are Fitted for the Two Subsets of Observed Maximum Likelihood Estimates, Separately

i	DGR. 3 Subset 1	DGR. 3 Subset 2	DGR. 4 Subset 1	DGR. 4 Subset 2
0	0.20878	0.21750	0.20953	0.24402
1	0.03405	-0.11653	0.03869	-0.31223
2	0.05388	0.13965	0.06056	0.45212
3	0.02235	-0.04411	0.02572	-0.21494
4			0.00055	0.02991

TABLE 5-3

Coefficients α_i's of the Resultant Polynomials of Degrees 3 and 4 , When They Are Fitted for the Five Subsets of Observed Maximum Likelihood Estimates, Separately

i		Subset 1	Subset 2	Subset 3	Subset 4	Subset 5
0	DGR. 3	-1.17587	0.56491	0.19259	2.67749	2.00465
1		-1.91141	0.91174	0.05255	-8.27070	-2.16327
2		-0.82751	0.74415	0.07557	8.59277	0.88624
3		-0.10701	0.20091	-0.25630	-2.81832	-0.13030
0	DGR. 4	15.22161	7.81272	0.18580	8.70223	25.44446
1		28.35231	32.93790	0.06613	-35.55154	-46.44038
2		19.74585	51.74863	0.34740	52.73598	31.81450
3		6.00195	34.99721	-0.38289	-33.19569	-9.60191
4		0.66909	8.61039	-1.28133	7.53819	1.07356

VI Approximation to the Standard Normal Density Function

We are to observe, in this section, how the method of moments and the least squares principle work in providing a polynomial to fit a relatively simple, symmetric function. For this purpose, we selected the standard normal density function, given by

$$(6.1) \qquad [2\pi]^{-1/2} \exp[-t^2/2] ,$$

where t denotes the random variable, for the function to be approximated. For the purpose of comparison, we are to consider another polynomial, which is known as Taylor's series. Let $h(t)$ be any function of the variable t which is n times differentiable. We can write for the Taylor's series

$$(6.2) \qquad h(t) = h(b) + (t - b)h'(b) + [(t - b)^2/2!]h''(b) + \dots + [(t - b)^{n-1}/(n - 1)!]h^{(n-1)}(b) + R_n(t) ,$$

where the remainder $R_n(t)$ is given by

$$(6.3) \qquad R_n(t) = [(t - b)^n/n!]\, h^{(n)}(\zeta)$$

and ζ is a value between t and the constant b. When the function $h(t)$ is infinitely differentiable and the remainder $R_n(t)$ uniformly converges to zero for some interval of t, the function $h(t)$ can be written as the power series such that

$$(6.4) \qquad h(t) = h(b) + (t - b)h'(b) + [(t - b)^2/2!]h''(b) + \dots\dots$$

It is well known that any normal density function, which can be written as

$$[2\pi]^{-1/2} \exp[-(t-\mu)^2/(2\sigma^2)] \quad , \tag{6.5}$$

has zero for all the odd moments about the mean, such that

$$\mu_{2r+1} = 0 \qquad r = 0, 1, 2, \ldots\ldots \tag{6.6}$$

and for the even moments we can write

$$\mu_{2r} = [\sigma^{2r}/2^r][(2r)!/r!] \tag{6.7}$$

(e.g., Kendall & Stuart, 1963, page 60). Calculating the even moments for $n(0,1)$ with $r = 0, 1, 2, 3, 4$ and 5, we obtain $\mu_0 = \mu_2 = 1$, $\mu_4 = 3$, $\mu_6 = 15$, $\mu_8 = 105$ and $\mu_{10} = 945$.

Using Hermite polynomials (e.g., Kendall & Stuart, 1963, page 155) and setting $b = 0$ in (6.4), we obtain for the Taylor's series for the standard normal density function

$$\begin{aligned} n(0,1) \doteq\ & 0.398942 - 0.199471t^2 + 0.0498677t^4 - 0.00831129t^6 \\ & + 0.00103891t^8 - 0.000103891t^{10} + \ldots\ldots \end{aligned} \tag{6.8}$$

It should be noted in (6.8) that all the exponentials to raise the variable t are even integers. In this polynomial, therefore, truncations at the degrees 3, 5, 7, etc., are the same as those at the degrees 2, 4, 6, etc.

Figure 6-1 presents the polynomials obtained by the method of moments using the interval of t , [-3.0, 3.0], with three different degrees, 3, 5 and 7, and those given by the Taylor's series truncated at the same degrees, respectively, together with the theoretical standard normal density function, which is given by (6.1).

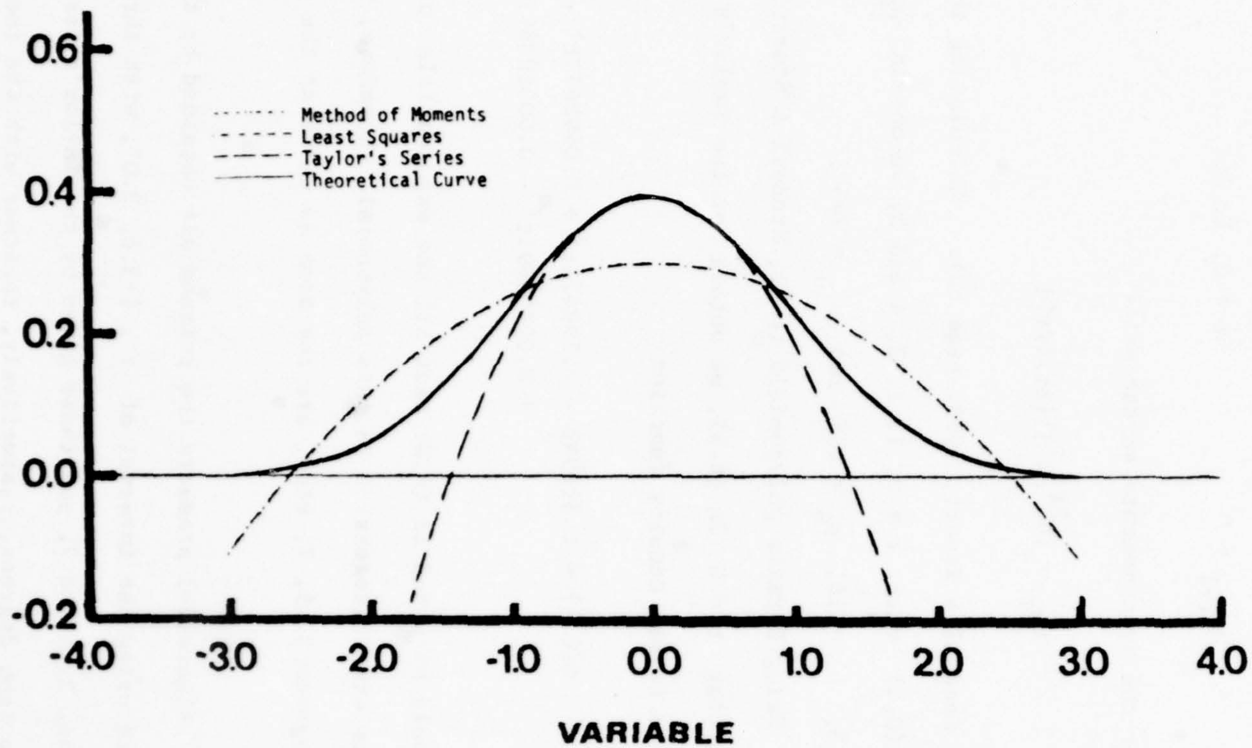

FIGURE 6-1

Comparison of the Polynomial Obtained by the Method of Moments and the Taylor's Series Polynomial Approximating the Standard Normal Density Function (Solid Curve). The Polynomials Are of Degree 3, Fitted for the Interval, [-3.0, 3.0].

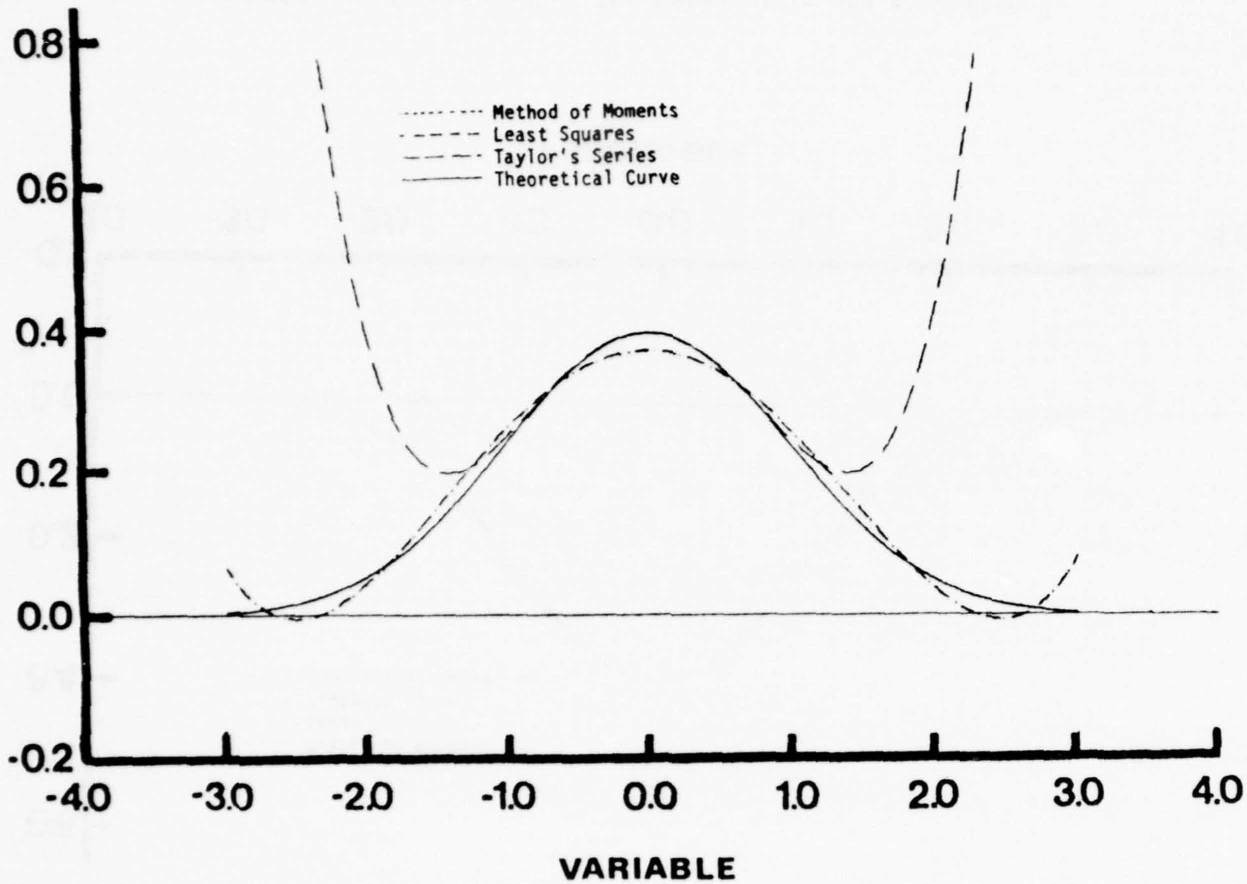

FIGURE 6-1 (Continued): The Polynomials Are of Degree 5.

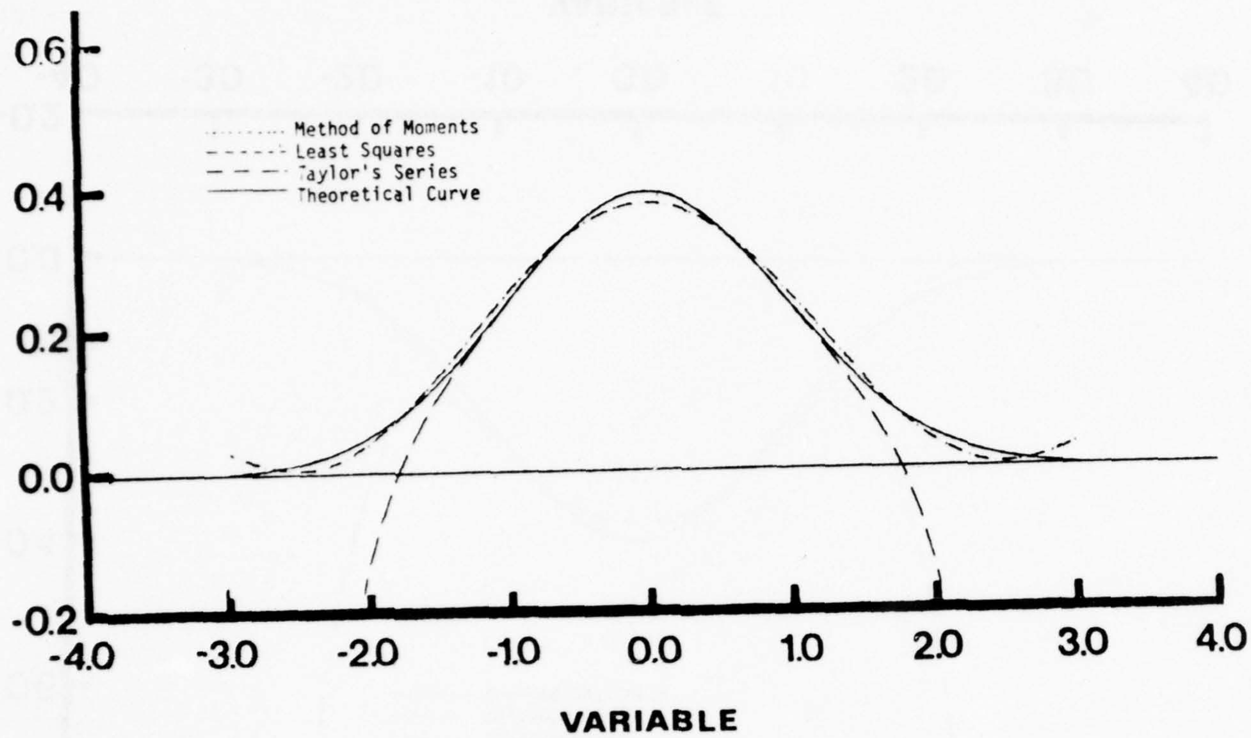

FIGURE 6-1 (Continued): The Polynomials Are of Degree 7.

It should be noted that, since the theoretical moments are used here, there is no possibility for the sampling fluctuations of the sample moments to affect the resultant polynomials obtained by the method of moments. Thus we can expect to obtain better fitting polynomials, as their orders increase. Comparison of the three curves in Figure 6-1 confirms this fact, and we can see the curve closest to the theoretical density function in the polynomial of degree 7. As was also expected, the Taylor's series polynomials approximate the theoretical curve very accurately around $t = 0$, and then depart from it for the values of t further from zero. The intervals of t for which the approximation is very high are, approximately, (-0.5, 0.5), (-0.8, 0.8) and (-1.2, 1.2) for the three Taylor's series polynomials, respectively, and (-1.5, 1.5) and (-1.7, 1.7) for the polynomials of degrees 9 and 11, although the last two are not shown here.

For the sake of comparison of the processes and results, we have also followed the process of the least square solution, which is outlined in section 3, to obtain the polynomials. Table 6-1 presents the coefficients of these polynomials of various degrees, together with those obtained by the method of moments. As is expected, these two sets of coeffients are practically identical, in all the cases. The curves for the polynomials by the method of moments in Figure 6-1 and the other figures presented later in this section, therefore, also represent those obtained through the process of the least squares solution.

Figure 6-2 presents the resultant polynomials of degrees 3, 5 and 7 obtained by the method of moments, using the intervals, [-4.0, 4.0], [-6.0, 6.0], [-8.0, 8.0] and [-10.0, 1.0], respectively. We can see

TABLE 6-1

Coefficients α_i's of the Resultant Polynomials of Degrees 3, 5 and 7 Fitted to the Standard Normal Density Function by the Method of Moments (M.M.) and the Least Squares Solution (L.S.) Using Five Different Intervals

Interval		[-3.0, 3.0]		[-4.0, 4.0]		[-6.0, 6.0]		[-8.0, 8.0]		[-10.0, 10.0]	
Method		M.M.	L.S.	M.M.	L.S.	M.M.	L.S.	M.M.	L.S.	M.M.	L.S.
0	DGR. 3	0.305556	0.305556	0.251953	0.251953	0.178819	0.178819	0.136963	0.136963	0.110625	0.110625
1		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2		-0.046296	-0.046296	-0.023804	-0.023804	-0.007957	-0.007957	-0.003490	-0.003490	-0.001819	-0.001819
3		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0	DGR. 5	0.373264	0.373262	0.332909	0.332908	0.257840	0.257838	0.204381	0.204380	0.167800	0.167798
1		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2		-0.121528	-0.121526	-0.074401	-0.074400	-0.029907	-0.029907	-0.014024	-0.014024	-0.007536	-0.007536
3		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4		0.009752	0.009752	0.003689	0.003689	0.000711	0.000711	0.000192	0.000192	0.000067	0.000067
5		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0	DGR. 7	0.383713	0.383702	0.373628	0.373613	0.315777	0.315749	0.261143	0.261138	0.219102	
1		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	
2		-0.145909	-0.145886	-0.127[illegible]44	-0.127828	-0.063704	-0.063690	-0.032649	-0.032648	-0.018310	
3		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	
4		0.017879	0.017872	0.013710	0.013707	0.003528	0.003527	0.001065	0.001065	0.000390	
5		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	
6		-0.000662	-0.000662	-0.000459	-0.000459	-0.000057	-0.000057	-0.000010	-0.000010	-0.000002	
7		0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	

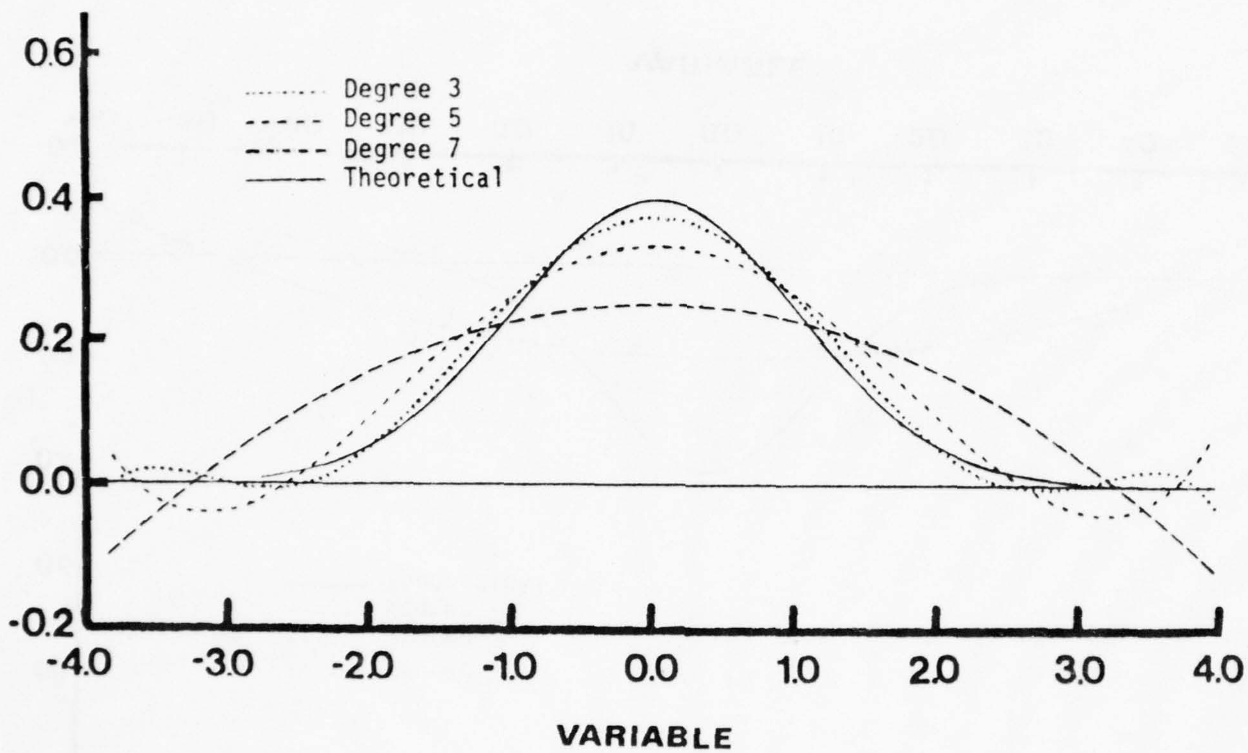

FIGURE 6-2

Comparison of the Three Polynomials of Degrees 3, 5 and 7 Obtained by the Method of Moments Approximating the Standard Normal Density Function. The Curves Are Fitted for the Interval [-4.0, 4.0].

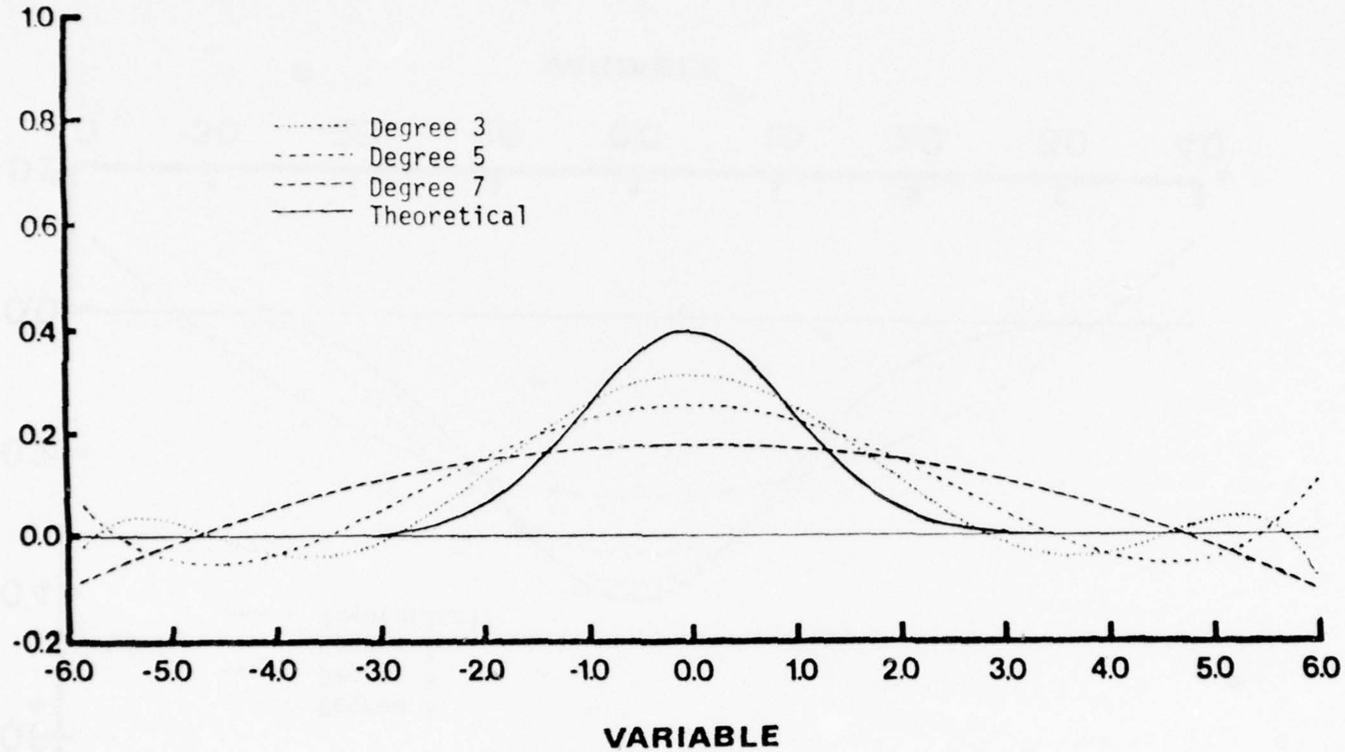

FIGURE 6-2 (Continued): The Curves Are Fitted for the Interval [-6.0, 6.0].

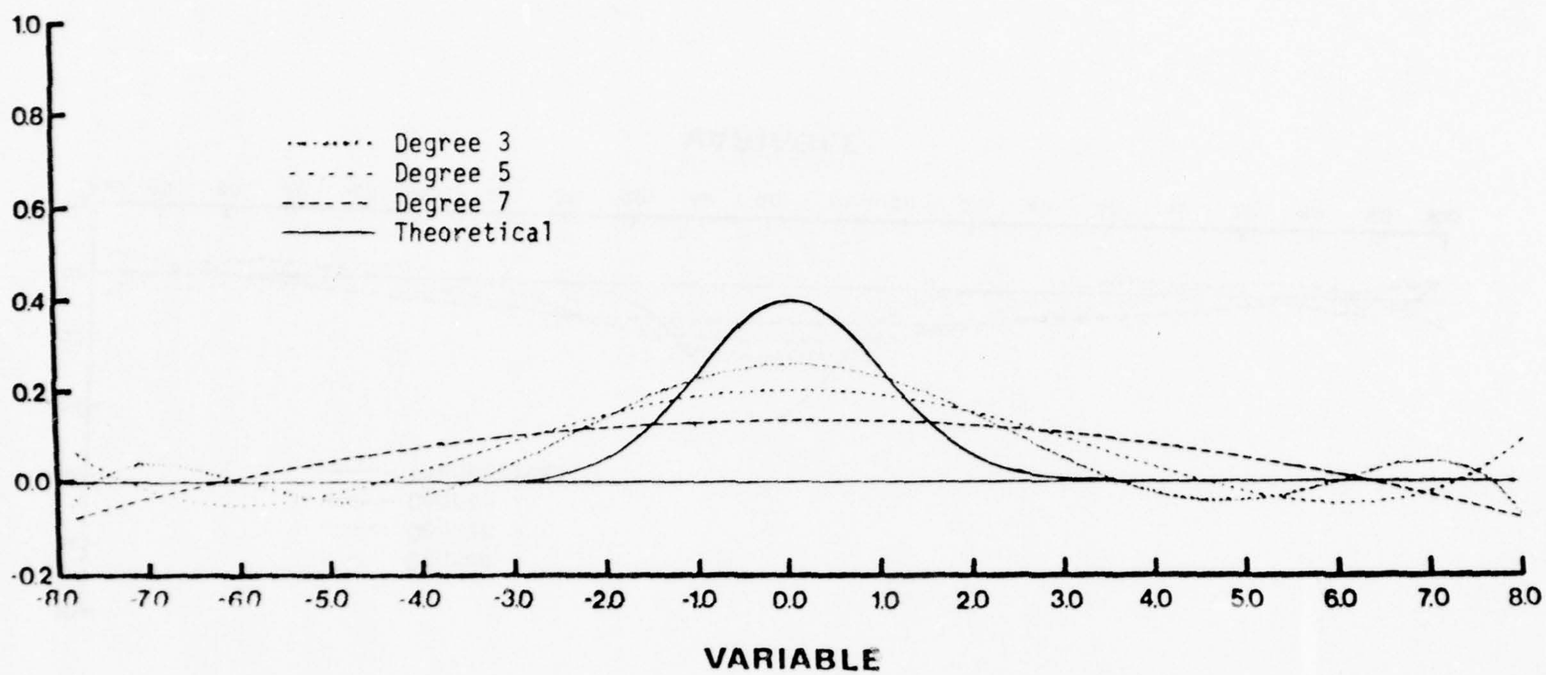

FIGURE 6-2 (Continued): The Curves Are Fitted for the Interval [-8.0, 8.0].

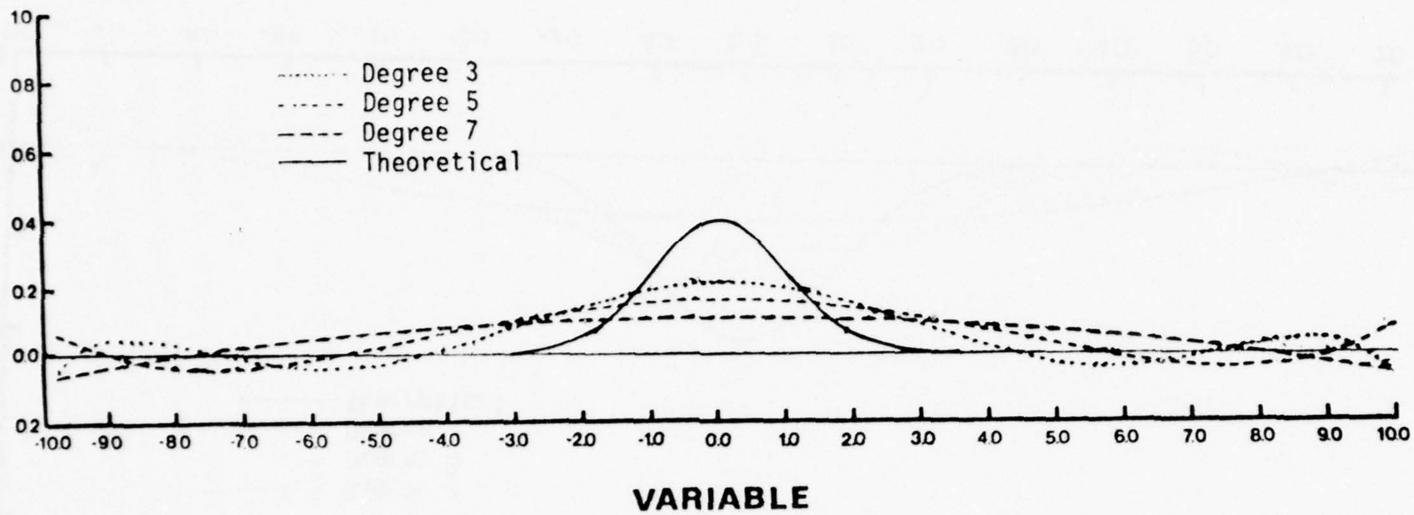

FIGURE 6-2 (Continued): The Curves Are Fitted for the Interval [-10.0, 10.0].

that, in all cases, the fit of the polynomial grows better as the degree of the polynomial increases.

Since the standard normal density function provides us with positive densities for the entire range of t such that

$$(6.9) \qquad -\infty < t < \infty ,$$

it is a matter of convenience where to truncate the interval of t . We notice in Figures 6-1 and 6-2, however, that the interval used for the method of moments, or the least squares solution, affects the resultant polynomial to a great extent. To observe it more closely, Figure 6-3 shows the comparison of the four curves obtained by using the intervals, [-4.0, 4.0], [-6.0, 6.0], [-8.0, 8.0] and [-10.0, 10.0], respectively, for each of the degrees, 3, 5 and 7. (The result from the interval, [-3.0, 3.0], is excluded in each case, since it is close to that from [-4.0, 4.0].)

We can see in Figure 6-3 that, in each case, the polynomial obtained by using the interval [-4.0, 4.0] provides the best fit to the theoretical curve, and it grows worse as the interval becomes larger. In fact, the polynomials of degree 3 obtained from [-8.0, 8.0] and [-10.0, 10.0], for example, are so flat that they are by no means close to the standard normal density function.

The above result may look strange considering the fact that we used the same set of *theoretical moments in the method* of moments to produce each set of curves. And yet it is fully understandable if we consider the fact that these polynomials are produced *upon the least squares principle*. To be more specific, if a polynomial is obtained

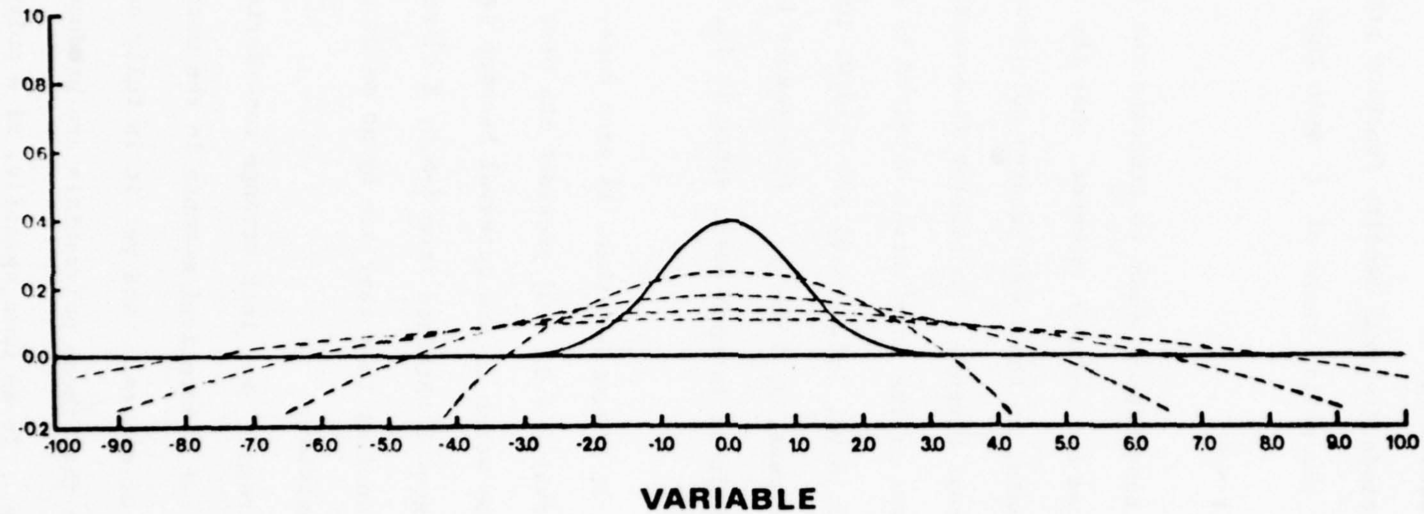

FIGURE 6-3

Comparison of the Four Polynomials Approximating the Standard Normal Density Function Fitted for the Four Different Intervals, [-4.0, 4.0], [-6.0, 6.0], [-8.0, 8.0] and [-10.0, 10.0]. The Polynomials Are of Degree 3.

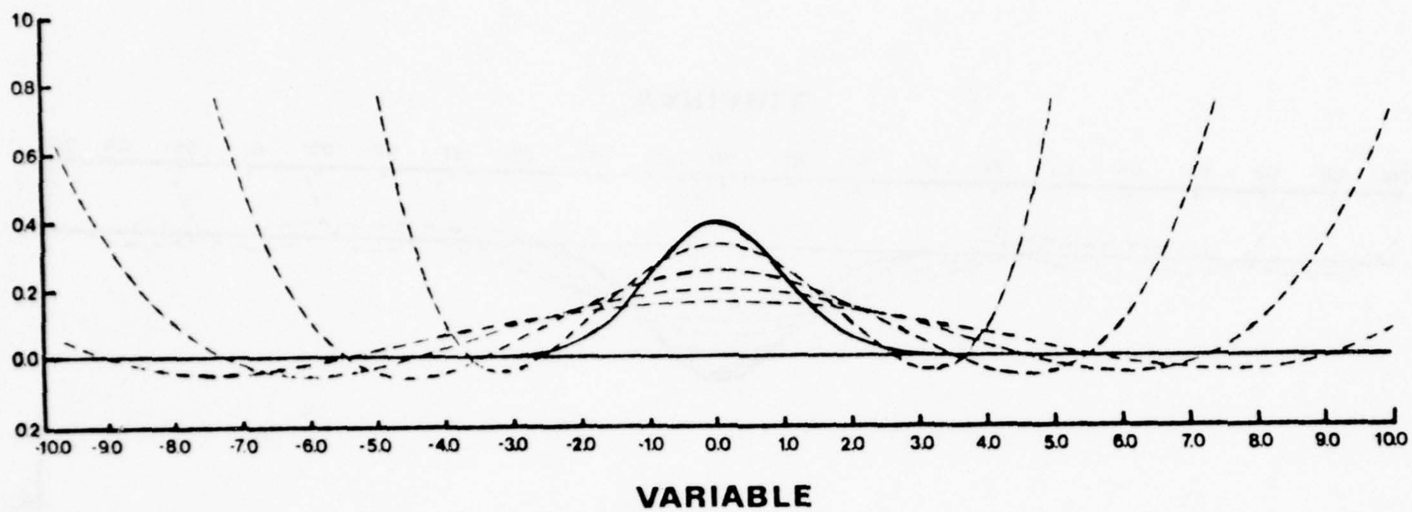

FIGURE 6-3 (Continued): The Polynomials Are of Degree 5.

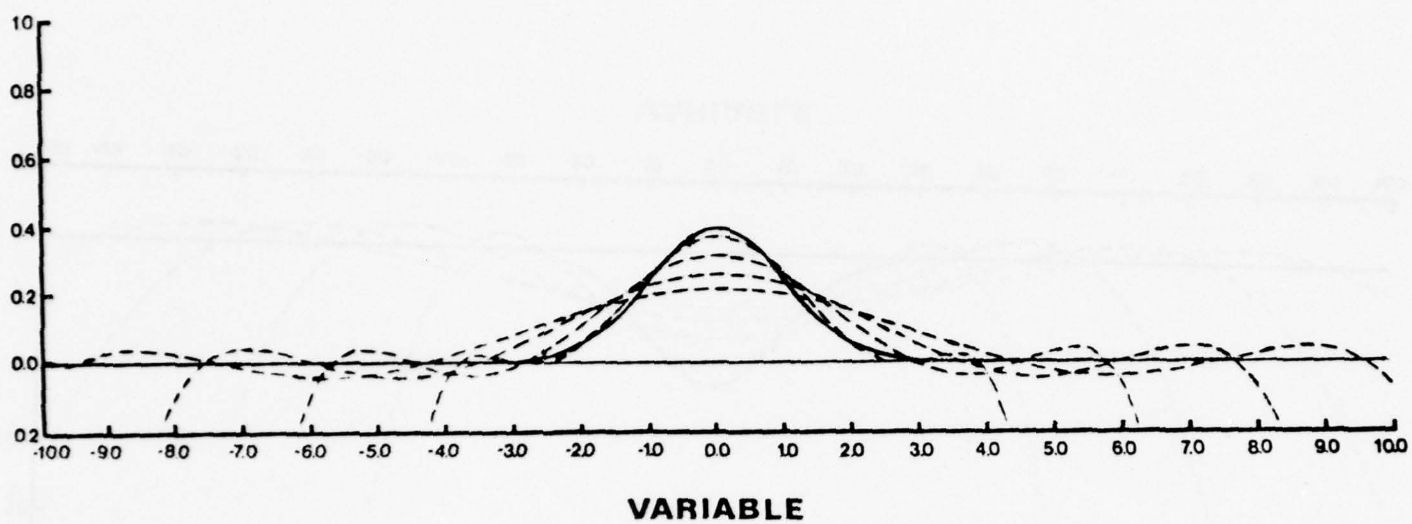

FIGURE 6-3 (Continued): The Polynomials Are of Degree 7.

from the interval, say, [-10.0, 10.0], then it will be the curve which is as close as possible to the concave part of the theoretical curve, *as well as to the parts of the theoretical curve which almost overlap the abscissa*, in the sense of the least squares solution, within the interval, [-10.0, 10.0]. The polynomial of degree 3 from this interval, therefore, must be very flat, and that of degree 7 must snake around the abscissa, as we can see in the first and third graphs of Figure 6-3.

The above observation gives us the warning that we should be extremely careful in selecting an optimal interval in using the method of moments, or the least squares solution. It will be advisable to ignore "tails" in certain occasions, or to divide the total set of data into several subsets and fit polynomials for each subset of data, if the sample size of each subset is still reasonably large.

We have seen the glimpse of warning in the preceding section that the use of sample moments of higher orders in the method of moments is likely to subject the resultant curve to large sampling fluctuations, especially when the sample size is small. The problem with moments of higher orders can be observed from a somewhat different angle. Table 6-2 presents five sets of the first through tenth moments computed for the standard normal distribution, using two different intervals, [-4.0, 4.0] and [-6.0, 6.0]. In these computations, Simpson's Method was used for integration, with a few different step widths. They are the moments about the mean, as well as the moments about the origin, since the mean of the standard normal distribution is zero.

We can see in Table 6-2 that, when we used the interval [-4.0, 4.0] for the computation of these moments, the resulting even moments are

TABLE 6-2

First through Tenth Moments About the Mean of the Standard Normal Density Function Using Two Different Intervals and Different Step Widths.

Step Width	Order of Moments	Interval (-4.0, 4.0)	Interval (-6.0, 6.0)
0.005	1	0.00000	0.00000
	2	0.99893	1.00000
	3	0.00001	0.00002
	4	2.97958	3.00006
	5	0.00003	0.00031
	6	14.62273	15.00058
	7	0.00026	0.00412
	8	97.95878	104.99695
	9	0.00235	0.05211
	10	811.27230	944.57312
0.025	1	0.00000	0.00000
	2	0.99890	1.00000
	3	0.00000	0.00000
	4	2.97909	2.99999
	5	0.00001	0.00001
	6	14.61505	14.99966
	7	0.00008	0.00008
	8	97.83744	104.98701
	9	0.00053	0.00053
	10	809.34313	944.51155
0.050	1		0.00000
	2		1.00000
	3		0.00000
	4		2.99996
	5		0.00001
	6		14.99890
	7		0.00010
	8		104.96610
	9		0.00077
	10		943.93550

not so close to the true values, which are 1, 3, 15, 105 and 945, respectively. In particular, the two tenth moments are approximately 135 less than the true value, 945. The result may be, at a first glance, puzzling, considering the negligibly small value of probability assigned to the whole range of t outside the interval [-4.0, 4.0], which is approximately 0.000065 . We must note, however, that the tenth power of 4.0, or -4.0, is as large as 1,048,576 , and those large absolute values outside the interval, [-4.0, 4.0], should certainly affect the result for a moment whose order is as large as ten. We can see in the same table that the values of the eighth moment, and even those of the sixth, are substantially less than the true values, when the interval [-4.0, 4.0] is used. When we use the other interval, [-6.0, 6.0], however, the values of the moments of higher orders are much closer to the true values, regardless of the step width used in Simpson's Method. Among the three sets of results using this other interval, there is a tendency that the accuracy of the computation of the even moments increases with the decrease in step width. This tendency does not appear for the odd moments, however, and, in total, the result of the interval [-6.0, 6.0] with the step width of 0.025 provides use with the best accuracy of computation.

The above observation about the accuracy of computation gives us an idea of how the truncation of the variable range can affect the values of moments of higher orders. Although it was observed in a different context, the same logic can be applied for the influence of the sampling fluctuations on the sample moments of higher orders,

i.e., possibilities of obtaining values which are so different from the population parameters because of the lack of, or the existence of, a few largely deviated observations.

VII Comparison of the Actual Processes of the Method of Moments and the Least Squares Solution

We have seen in the preceding sections that the two processes of the method of moments and of the least square solution provide us with identical polynomials, except for the rounding errors, as they should since they are the same polynomial in theory, with respect to the standard normal density function. Because of the practical restriction involved in computer work, however, the same is not always true in other occasions. It has been observed in the authors' experience that the least squares solution tends to give us more trouble, since it includes the inversion of the transformation matrix A, as is described in Section 3. In fact, as long as we use polynomials of degree 7 or less, we have not come across any troubles with the method of moments, and once the appropriate program has been written for the method, the process is very simple.

This fact has a great deal of benefit to researchers, since the method of moments for fitting a polynomial is no longer restricted to a method of estimating a density function or a frequency function, but can be used as the least squares solution for *any function*.

As an example of fitting a curve to a function which is not a density function, we shall use the standard normal distribution function given by

$$(7.1) \qquad (2\pi)^{-1/2} \int_{-\infty}^{t} \exp(-u^2/2)\, du \quad ,$$

and treat it as if it were an empirically obtianed function. Also, as we have done with the standard normal density function, we shall compare the result of the method of moments, or the least squares solution, with the polynomial obtained by Taylor's series. We set $b = 0$ in

the result of the method of moments, or the least squares solution, with the polynomial obtained by Taylor's series. Thus from (6.2), (6.6) and (6.7), using Hermite polynomials (Kendall and Stuart, 1963, pages 155-156), we can write

$$N(0,1) \doteq 0.500000 + 0.398942t - 0.0664903t^3 + 0.00997355t^5 - 0.00118732t^7 + 0.000115434t^9 - 0.00000944465t^{11} + \ldots \tag{7.2}$$

Figure 7-1 presents the polynomials of degrees 3, 4, 5, 6 and 7 obtained by the method of moments, and those obtained by the least squares solution, following the separate procedures described in Sections 4 and 3, respectively. In the same figure, also presented are the polynomials obtained from the Taylor's series, which are truncated to provide us with the polynomials of degrees 3, 4, 5, 6 and 7, respectively. The interval of the variable t, which is used both in the method of moments and in the least squares solution, is [-3.0, 3.0], for which the standard normal distribution function strictly increases, approximately, from zero to unity.

We can see that, in this example, for each degree, the polynomials obtained by the two methods, which are plotted by a dotted line and a shorter dashed line, respectively, are almost indistinguishable from each other. We can also see that the fit of these two curves to the theoretical normal distribution function, which is drawn by a solid line, is as good as that of the polynomial obtained from Taylor's series for values of t close to zero, if the degree of the polynomial is five or greater, and outside of this subinterval the

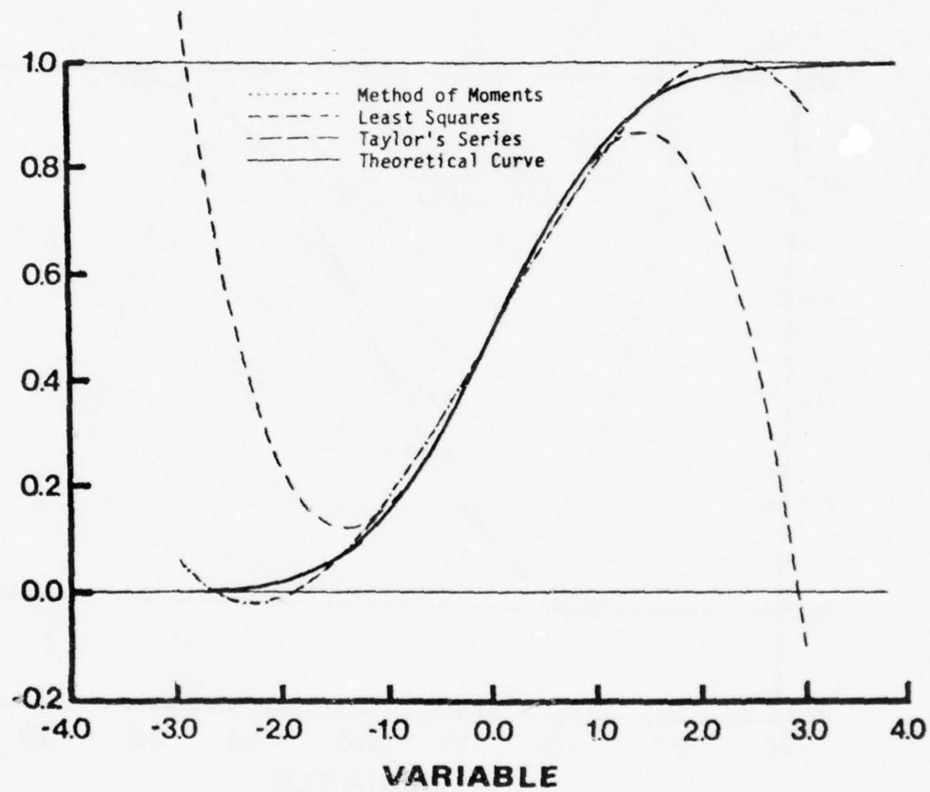

FIGURE 7-1

Polynomials Obtained by the Method of Moments and the Least Squares Solution, with the Interval, [-3.0, 3.0], and the Taylor's Series, Approximating the Standard Normal Distribution Function. They Are of Degree 3.

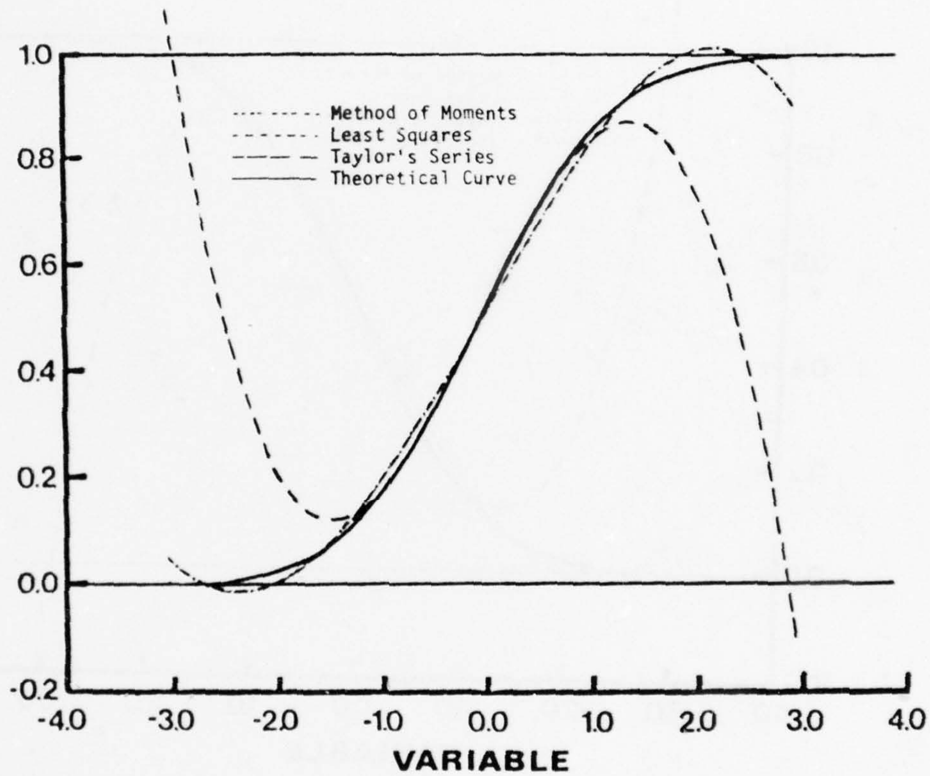

FIGURE 7-1 (Continued): Polynomials Are of Degree 4 .

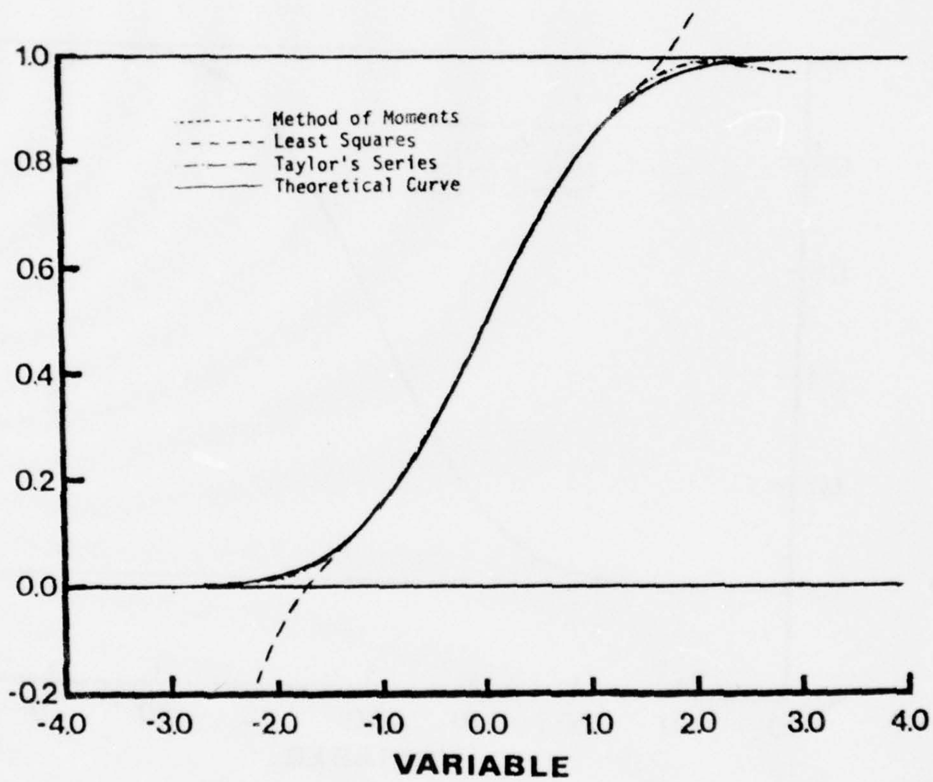

FIGURE 7-1 (Continued): Polynomials Are of Degree 5 .

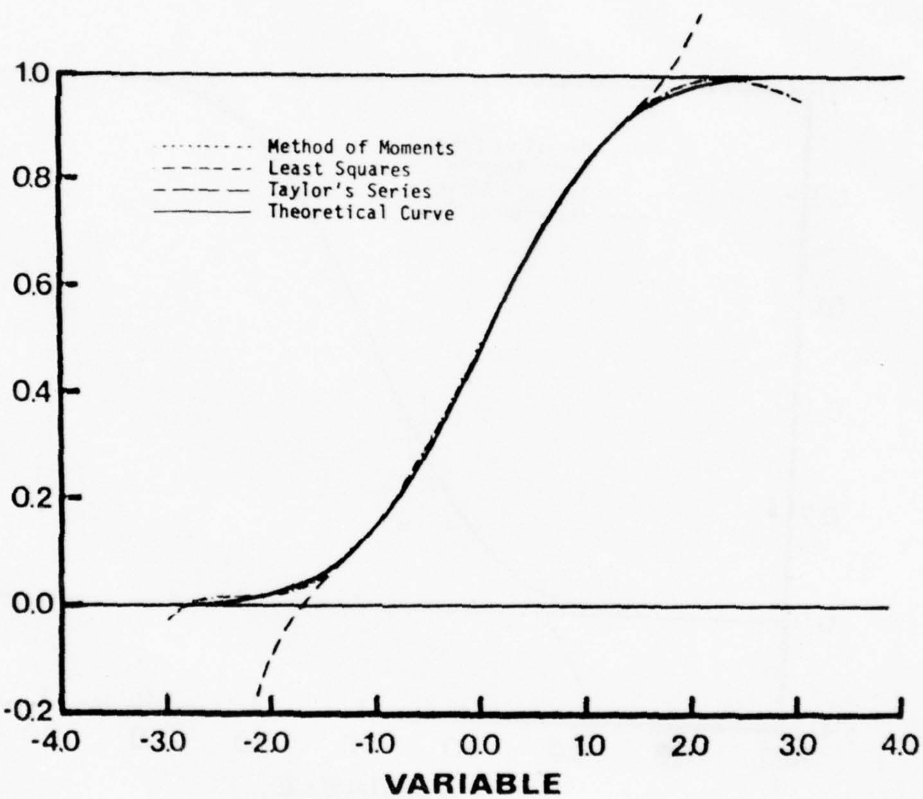

FIGURE 7-1 (Continued): Polynomials Are of Degree 6 .

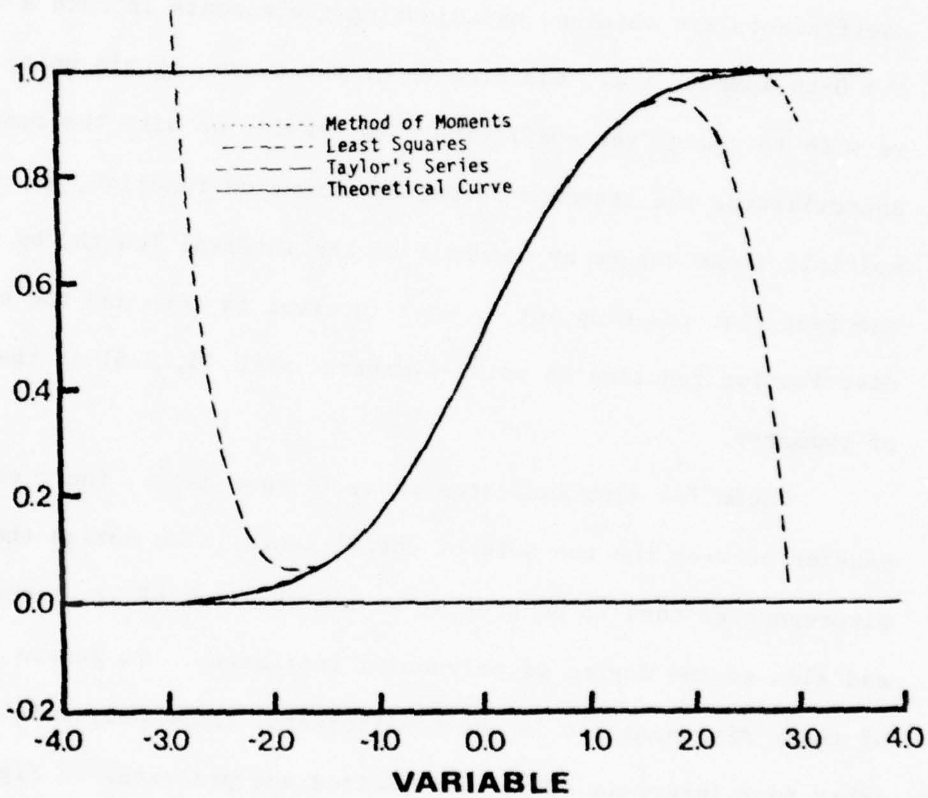

FIGURE 7-1 (Continued): Polynomials Are of Degree 7 .

fit is substantially better than that of the Taylor's series polynomials. The coefficients of the two polynomials are shown in Table 7-1, as well as those obtained by using four other intervals of t , [-4.0, 4.0], [-6.0, 6.0], [-8.0, 8.0] and [-10.0, 10.0]. These coefficients are obtained by adjusting the moments in such a way that the 0-th moment, i.e., the area under the curve, equals unity. If we wish to change the coefficients to provide us with the real curve approximating the standard normal distribution function, we can simply multiply those values by one half of the interval length, by virtue of the fact that the midpoint of each interval is zero and the normal distribution function is point-symmetric with (0, 0.5) as the point of symmetry.

Table 7-1 also indicates that, in some cases, there exist discrepancies between the two sets of coefficients. We notice that these discrepancies tend to be greater as the interval of t becomes greater, and also as the degree of polynomial increases. To pursue the effect of these discrepancies in the coefficients, these polynomials for the other four intervals of t are plotted and presented in Figures 7-2 through 7-5, in the same manner as Figure 7-1.

It is rather astonishing to see large discrepancies between the two curves obtained by the method of moments procedure and the least squares solution procedure, respectively, especially for the intervals, [-6.0, 6.0] or above, and for the polynomials of degrees 6 and 7 . This is explained by the fact that small discrepancies in the values of coefficients for higher powers may affect the resulting curves drastically. It is interesting to note that the curves obtained by

TABLE 7-1

Coefficients α_i's of the Resultant Polynomials of Degrees 3, 4, 5, 6 and 7 Fitted to the Standard Normal Distribution Function by the Method of Moments (M.M.) and the Least Squares Solution (L.S.) Using Five Different Intervals. Coefficients Are Adjusted to Make the Area Under Each Curve Unity.

Interval		[-3.0, 3.0]		[-4.0, 4.0]		[-6.0, 6.0]		[-8.0, 8.0]		[-10.0, 10.0]	
Method		M.M.	L.S.	M.M.	L.S.	M.M.	L.S.	M.M.	L.S.	M.M.	L.S.
0	DGR. 3	0.168210	0.168210	0.125916	0.125916	0.083755	0.083755	0.062740	0.062740	0.050155	0.050155
1		0.113651	0.113651	0.072749	0.072749	0.035859	0.035859	0.020946	0.020946	0.013643	0.013643
2		-0.000514	-0.000514	-0.000172	-0.000172	-0.000035	-0.000035	-0.000011	-0.000011	-0.000005	-0.000005
3		-0.007378	-0.007378	-0.003016	-0.003016	-0.000725	-0.000725	-0.000246	-0.000246	-0.000104	-0.000104
0	DGR. 4	0.166666	0.166666	0.125176	0.125176	0.083472	0.083472	0.062590	0.062590	0.050062	0.050062
1		0.113651	0.113651	0.072749	0.072749	0.035859	0.035859	0.020946	0.020946	0.013643	0.013643
2		0.001201	0.001201	0.000291	0.000291	0.000044	0.000043	0.000012	0.000012	0.000005	0.000004
3		-0.007378	-0.007378	-0.003016	-0.003016	-0.000725	-0.000725	-0.000246	-0.000246	-0.000104	-0.000104
4		-0.000222	-0.000222	-0.000034	-0.000034	-0.000003	-0.000002	-0.000000	-0.000000	-0.000000	-0.000000
0	DGR. 5	0.166666	0.166666	0.125176	0.125176	0.083472	0.083472	0.062590	0.062590	0.050062	0.050062
1		0.126680	0.126682	0.087811	0.087812	0.047343	0.047344	0.028771	0.028772	0.019116	0.019119
2		0.001201	0.001201	0.000291	0.000291	0.000044	0.000043	0.000012	0.000012	0.000005	0.000004
3		-0.014133	-0.014134	-0.007409	-0.007409	-0.002213	-0.002214	-0.000816	-0.000818	-0.000359	-0.000359
4		-0.000222	-0.000222	-0.000034	-0.000034	-0.000003	-0.000002	-0.000000	-0.000000	-0.000000	-0.000000
5		0.000676	0.000676	0.000247	0.000247	0.000037	0.000037	0.000008	0.000010	0.000002	-0.000005
0	DGR. 6	0.168183	0.168181	0.126210	0.126209	0.084011	0.084010	0.062912	0.062908	0.050273	0.050088
1		0.126680	0.126682	0.087811	0.087812	0.047343	0.047344	0.028771	0.028772	0.019116	0.019119
2		-0.002339	-0.002335	-0.001066	-0.001065	-0.000271	-0.000273	-0.000093	-0.000100	-0.000040	-0.000054
3		-0.014133	-0.014134	-0.007409	-0.007409	-0.002213	-0.002214	-0.000816	-0.000818	-0.000359	-0.000359
4		0.000957	0.000956	0.000221	0.000220	0.000024	0.000024	0.000005	0.000008	0.000001	-0.000020
5		0.000676	0.000676	0.000247	0.000247	0.000037	0.000037	0.000008	0.000010	0.000002	-0.000005
6		-0.000096	-0.000096	-0.000012	-0.000012	-0.000001	0.000001	-0.000000	-0.000017	-0.000000	-0.000000
0	DGR. 7	0.168183	0.168181	0.126210	0.126209	0.084011	0.084010	0.062912	0.062908	0.050273	0.050088
1		0.137267	0.137294	0.098033	0.098050	0.056338	0.056358	0.035557	0.035454	0.024119	0.014024
2		-0.002339	-0.002335	-0.001066	-0.001065	-0.000271	-0.000273	-0.000093	-0.000100	-0.000040	-0.000054
3		-0.024718	-0.024745	-0.013158	-0.013166	-0.004462	-0.004462	-0.001770	-0.001784	-0.000810	-0.000238
4		0.000957	0.000956	0.000221	0.000220	0.000024	0.000024	0.000005	0.000008	0.000001	-0.000020
5		0.003264	0.003269	0.001038	0.001040	0.000175	0.000161	0.000041	0.000000	0.000012	-0.000001
6		-0.000096	-0.000096	-0.000012	-0.000012	-0.000001	0.000001	-0.000000	-0.000017	-0.000000	-0.000000
7		-0.000178	-0.000178	-0.000031	-0.000030	-0.000002	-0.000019	-0.000000	-0.000041	-0.000000	-0.000000

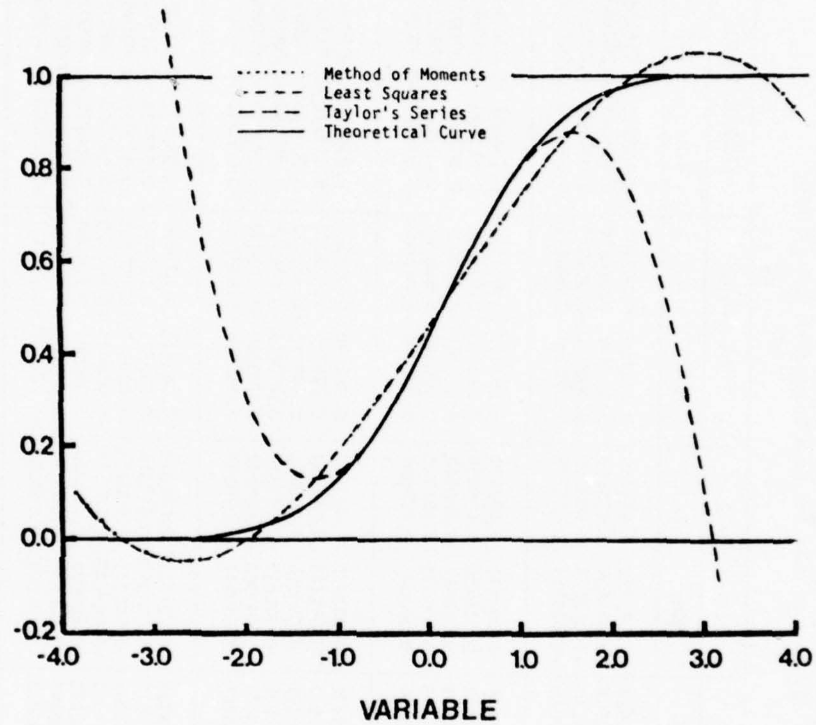

FIGURE 7-2

Polynomials Obtained by the Method of Moments and the Least Squares Solution, with the Interval, [-4.0, 4.0], and the Taylor's Series, Approximating the Standard Normal Distribution Function. They Are of Degree 3.

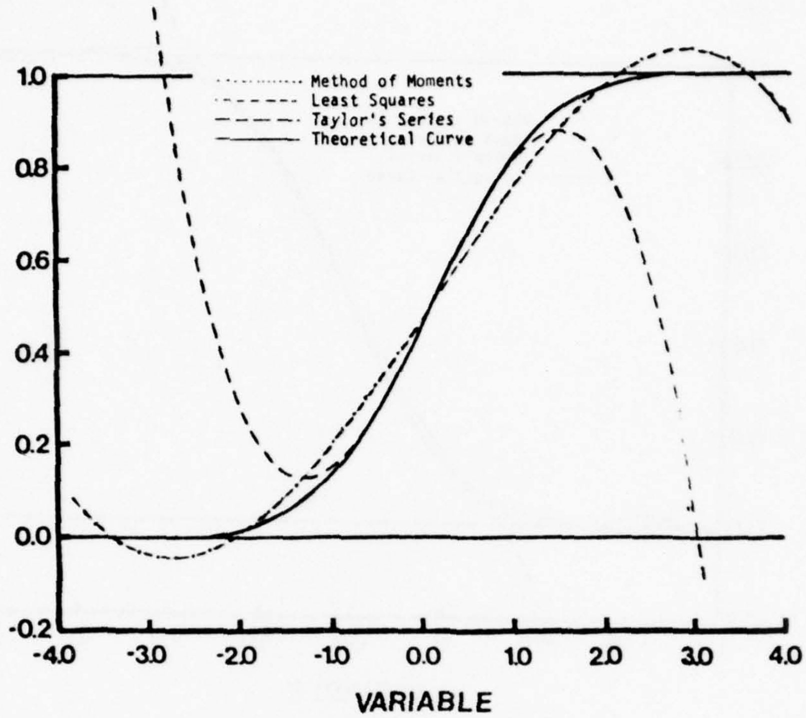

FIGURE 7-2 (Continued): Polynomials Are of Degree 4.

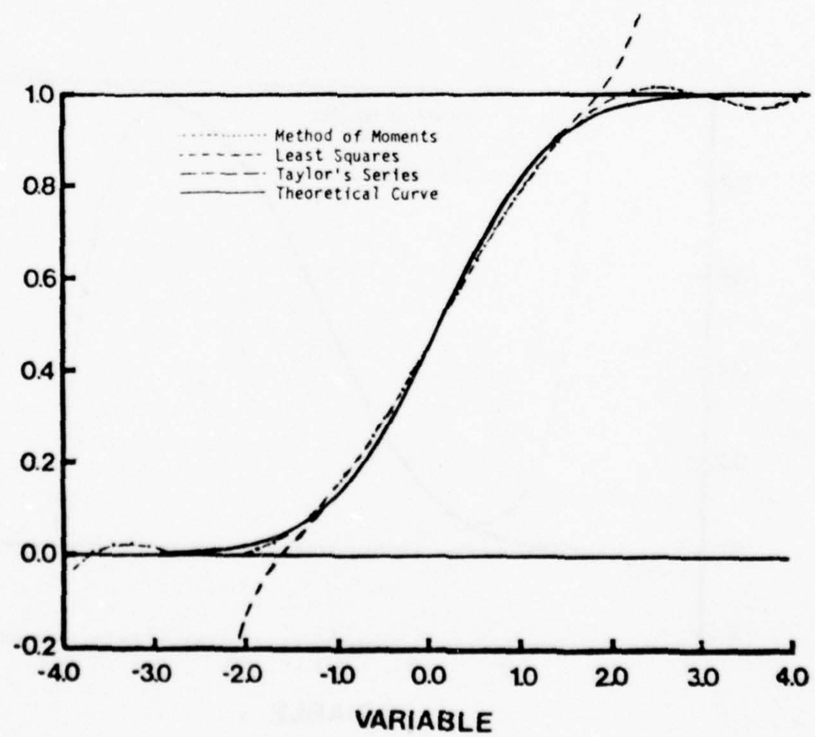

FIGURE 7-2 (Continued): Polynomials Are of Degree 5.

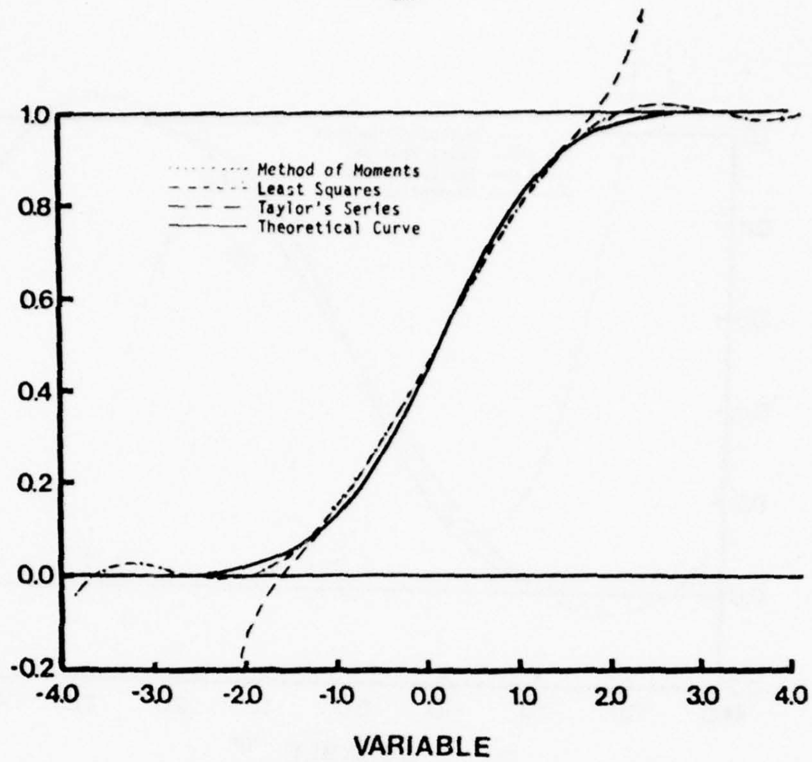

FIGURE 7-2 (Continued): Polynomials Are of Degree 6.

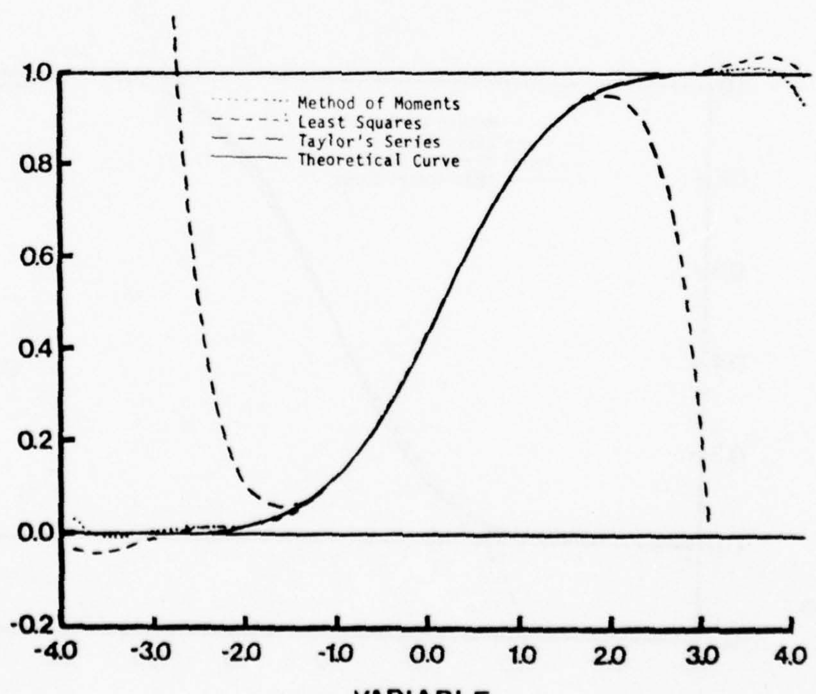

FIGURE 7-2 (Continued): Polynomials Are of Degree 7.

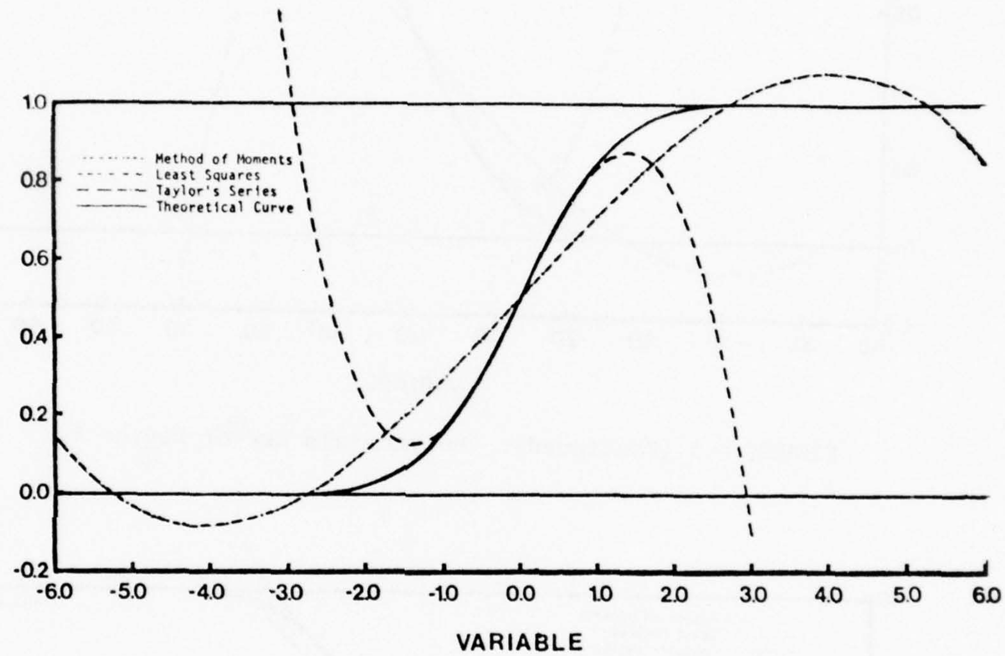

FIGURE 7-3

Polynomials Obtained by the Method of Moments and the Least Squares Solution, with the Interval, [-6.0, 6.0], and the Taylor's Series, Approximating the Standard Normal Distribution Function. They Are of Degree 3.

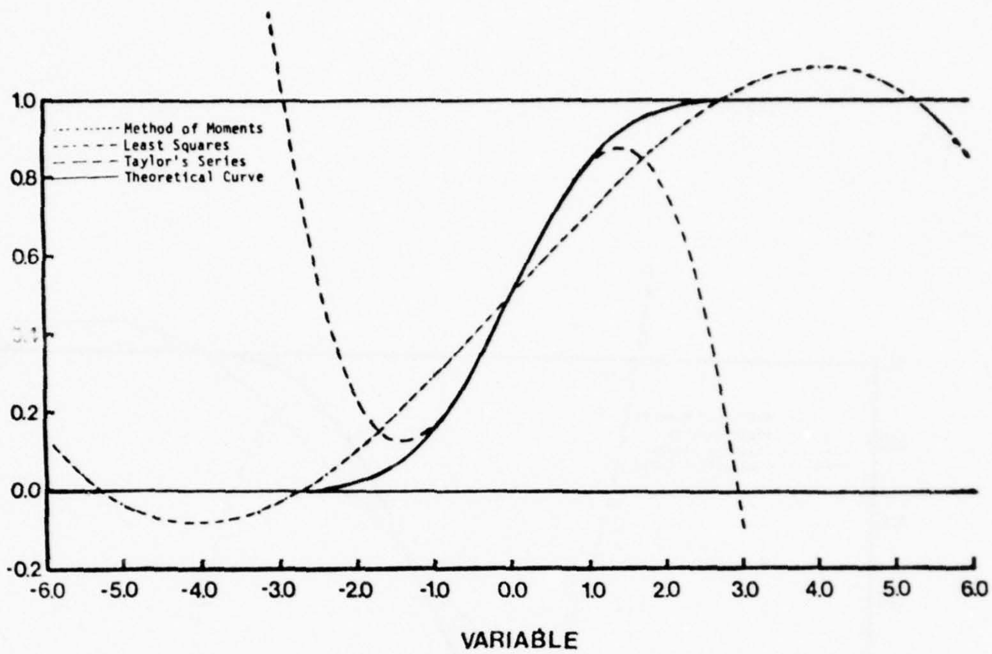

FIGURE 7-3 (Continued): Polynomials Are of Degree 4.

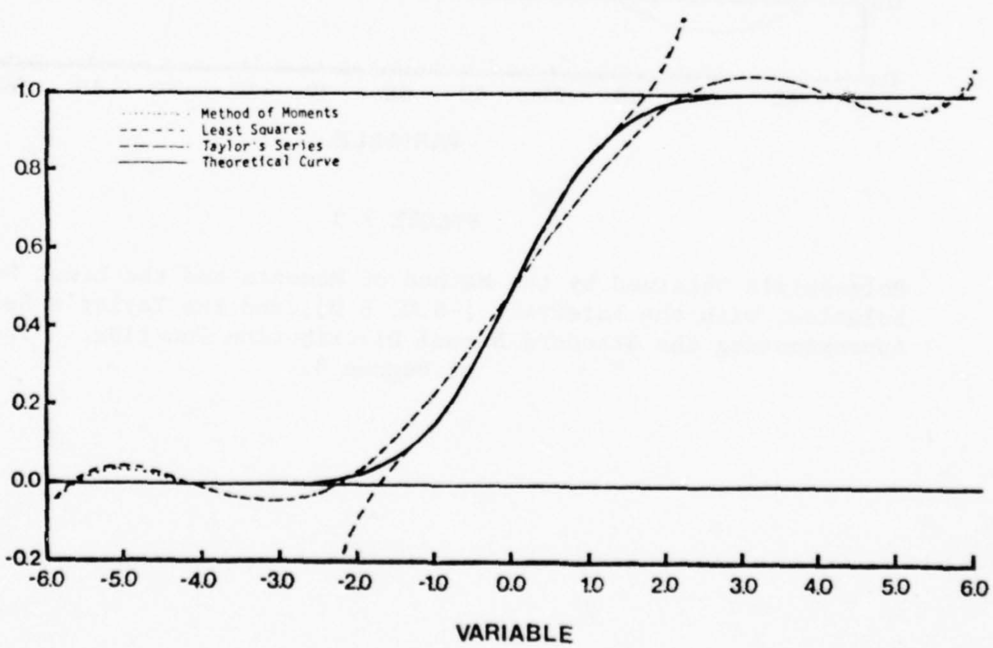

FIGURE 7-3 (Continued): Polynomials Are of Degree 5.

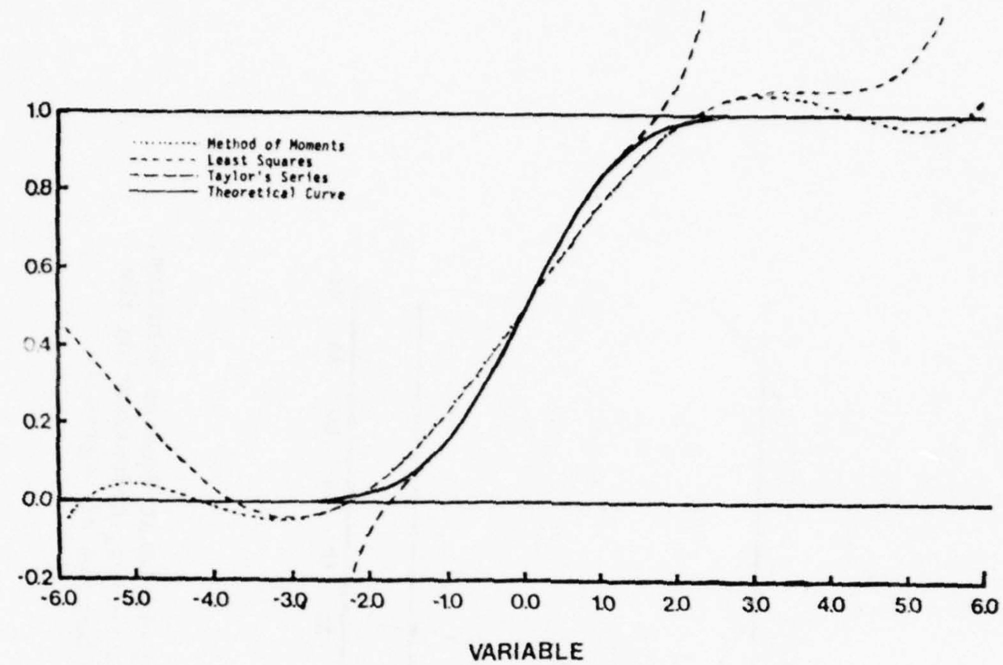

FIGURE 7-3 (Continued): Polynomials Are of Degree 6.

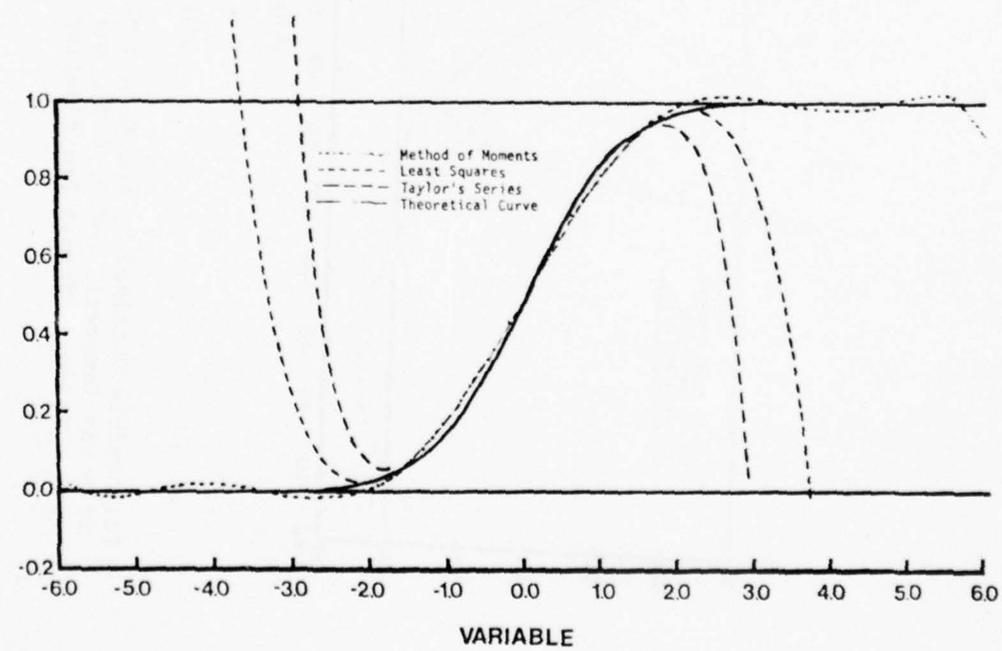

FIGURE 7-3 (Continued): Polynomials Are of Degree 7.

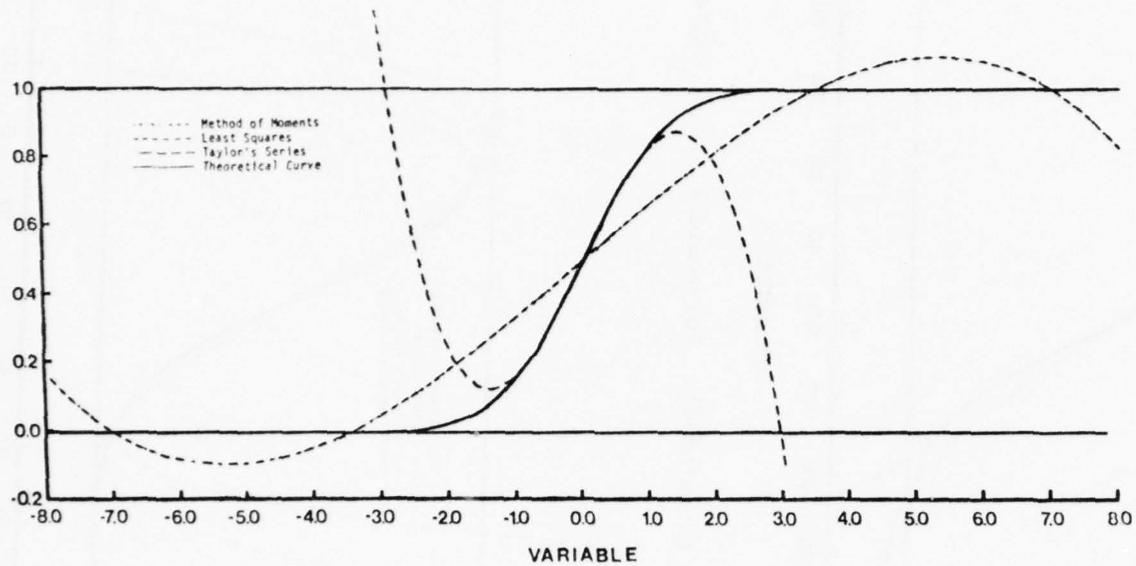

FIGURE 7-4

Polynomials Obtained by the Method of Moments and the Least Squares Solution, with the Interval, [-8.0, 8.0], and the Taylor's Series, Approximating the Standard Normal Distribution Function. They Are of Degree 3.

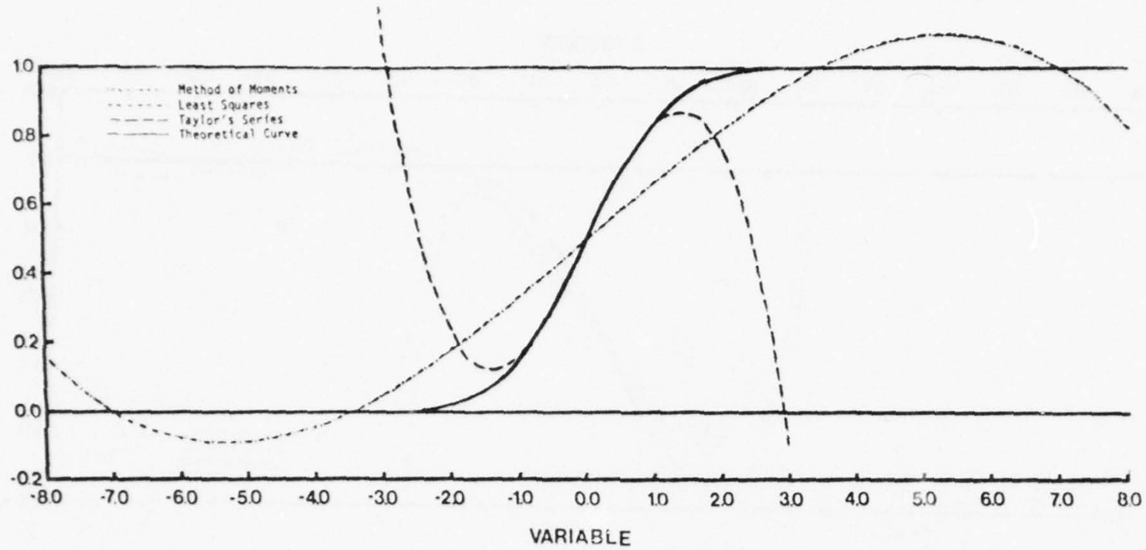

FIGURE 7-4 (Continued): Polynomials Are of Degree 4.

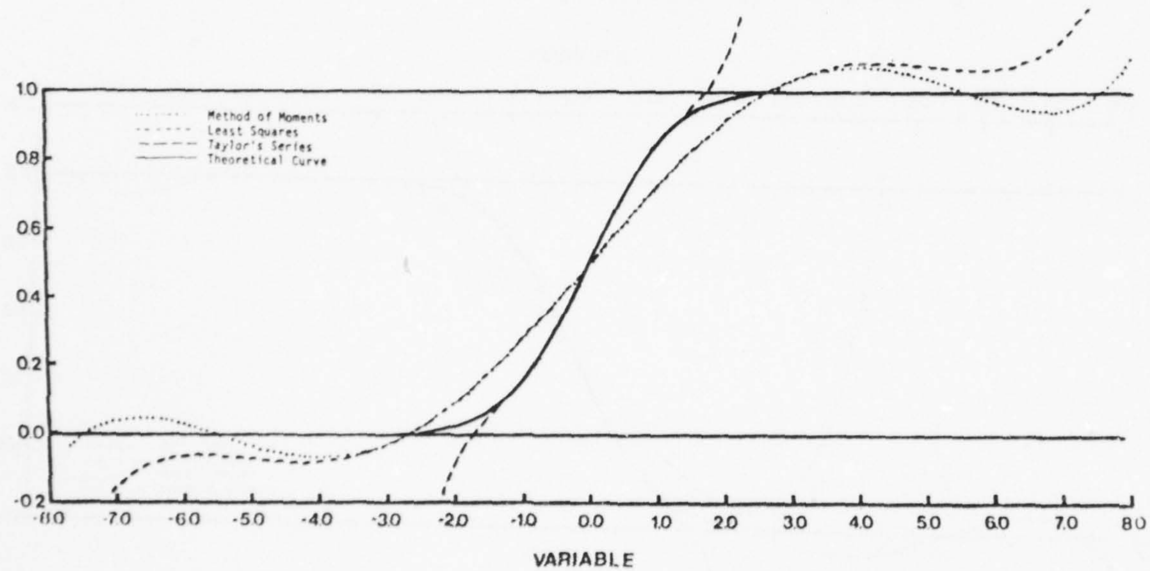

FIGURE 7-4 (Continued): Polynomials Are of Degree 5.

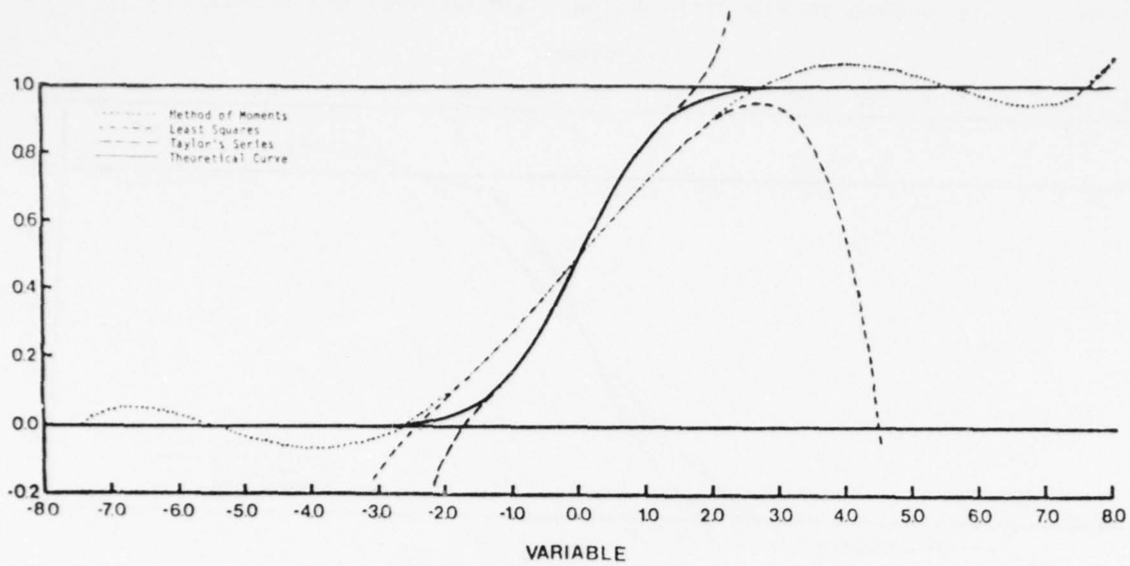

FIGURE 7-4 (Continued): Polynomials Are of Degree 6.

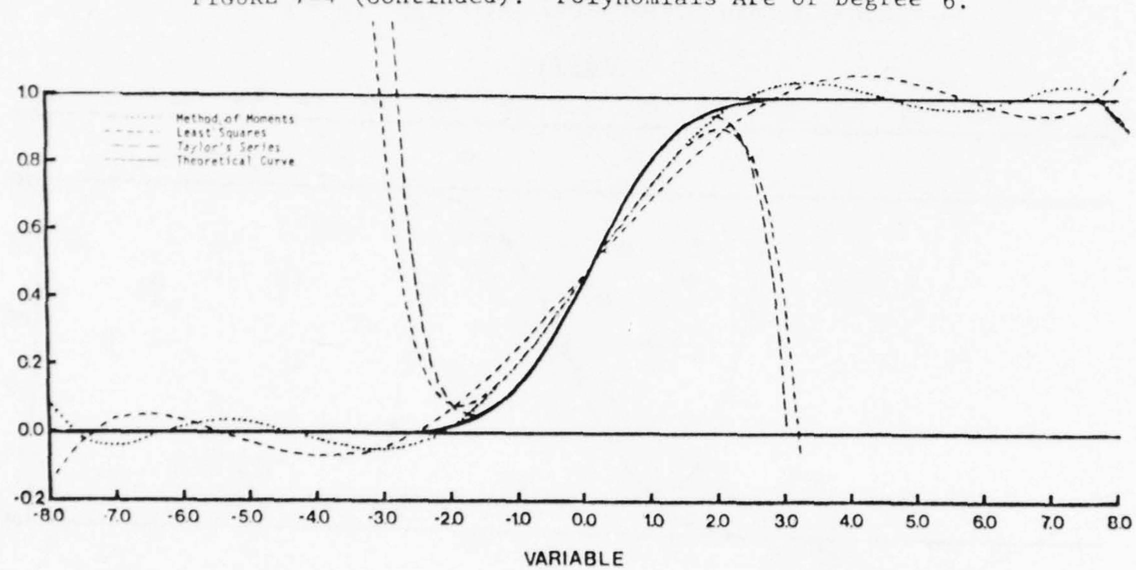

FIGURE 7-4 (Continued): Polynomials Are of Degree 7.

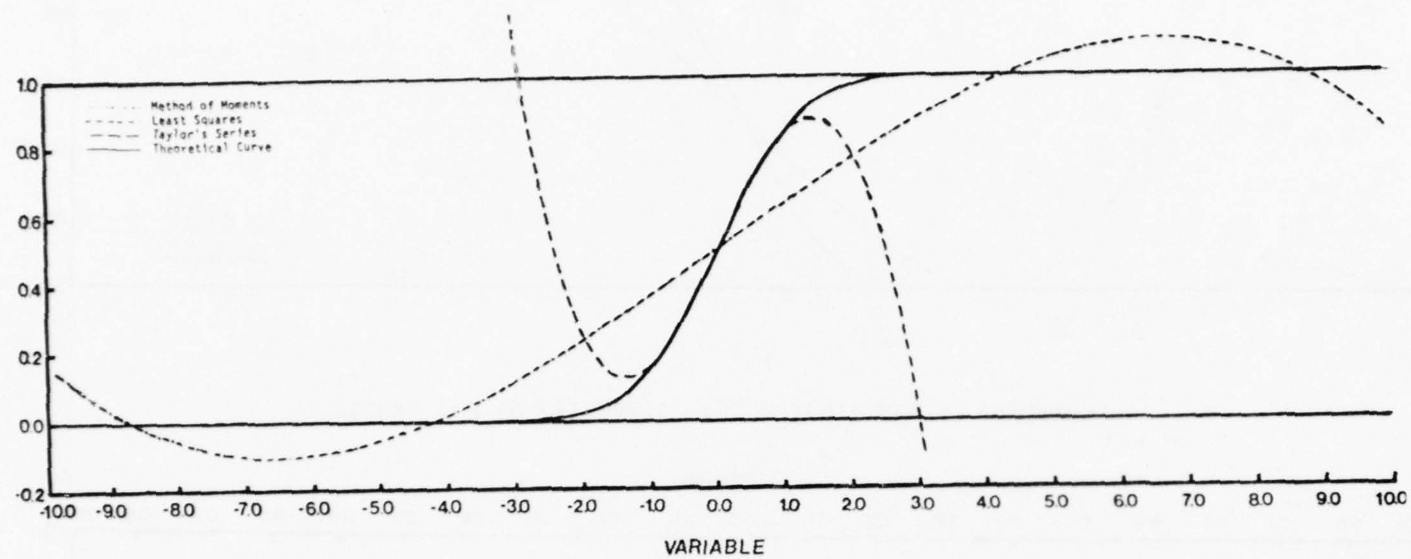

FIGURE 7-5

Polynomials Obtained by the Method of Moments and the Least Squares Solution, with the Interval, [-10.0, 10.0], and the Taylor's Series, Approximating the Standard Normal Distribution Function. They Are of Degree 3.

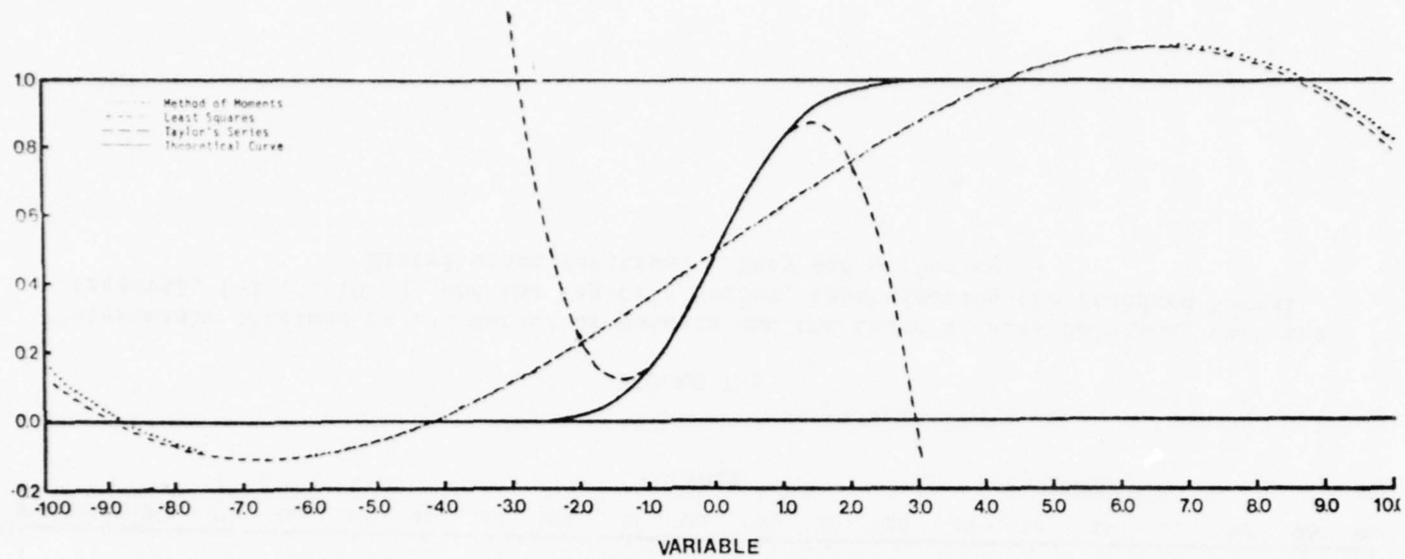

FIGURE 7-5 (Continued): Polynomials Are of Degree 4.

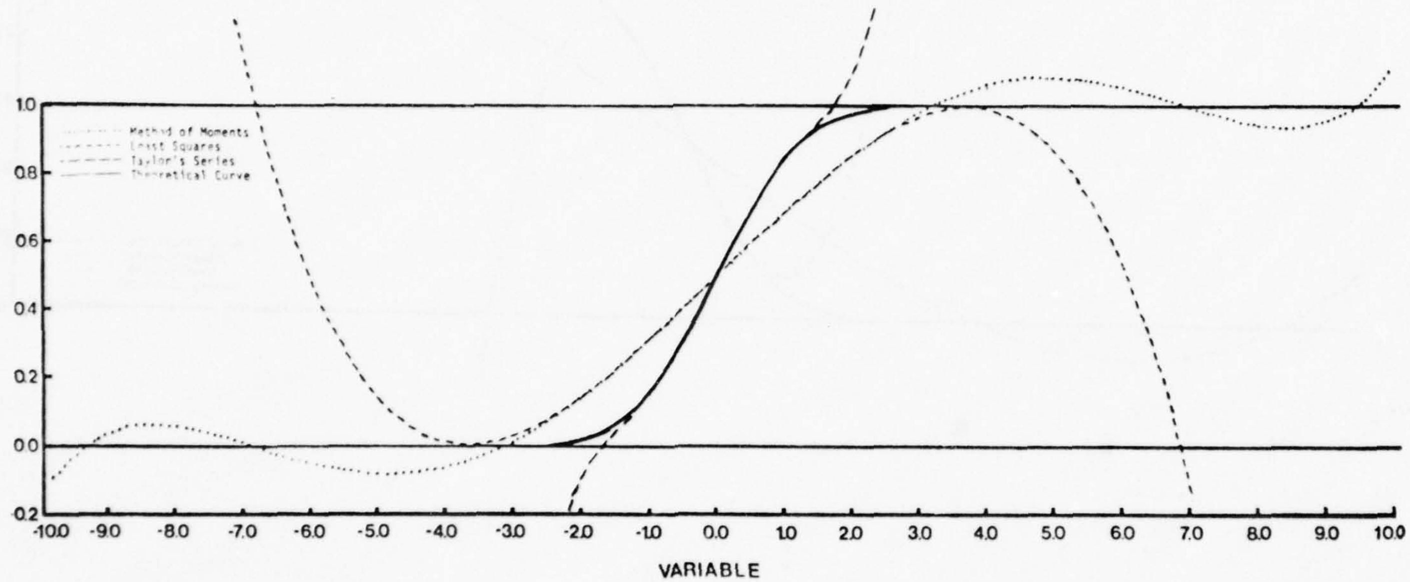

FIGURE 7-5 (Continued): Polynomials Are of Degree 5.

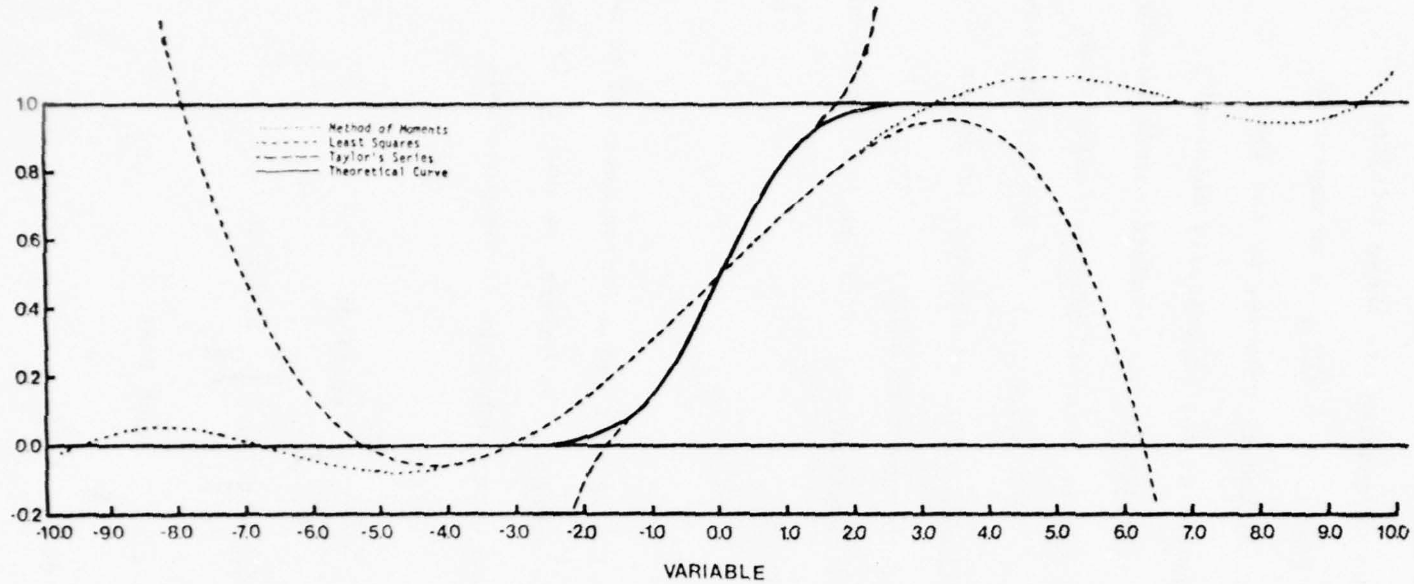

FIGURE 7-5 (Continued): Polynomials Are of Degree 6.

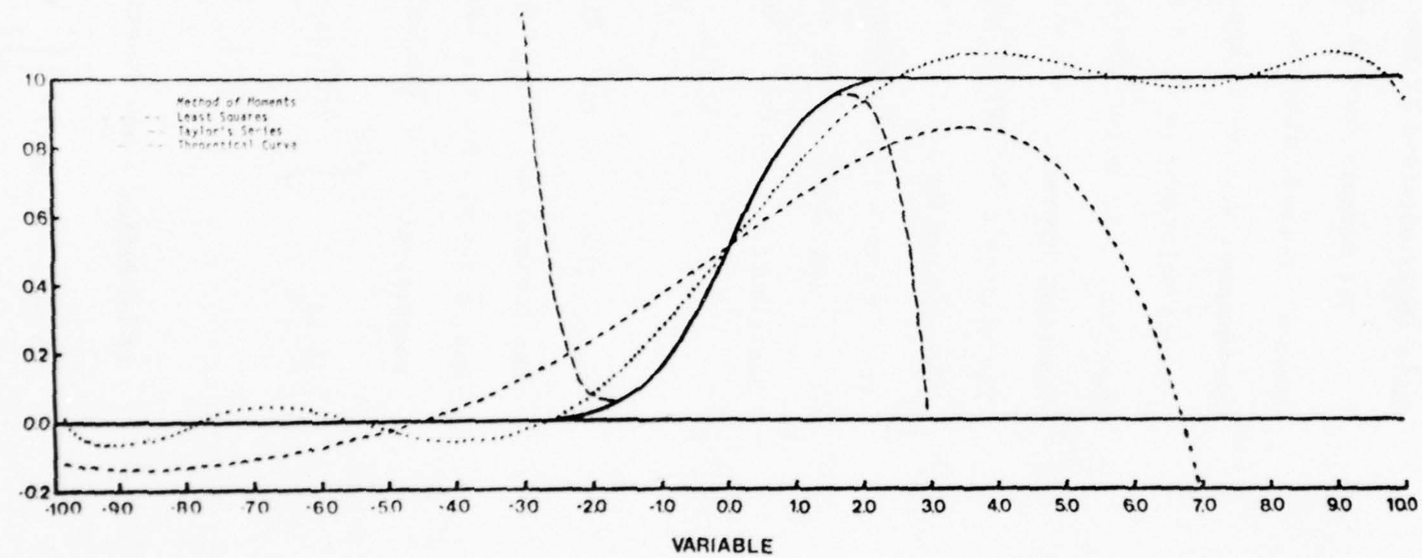

FIGURE 7-5 (Continued): Polynomials Are of Degree 7.

the method of moments procedure demonstrate the least squares principle, while those obtained by the least squares solution do not.

The moments used in the two procedures are those obtained by Simpson's Method, with the step width of 0.025, as we expect to use whenever we wish to apply the method of moments, or the least squares solution, to fit a polynomial to any empirically obtained function. Since our choice of function is the standard normal density function, however, these moments can be obtained mathematically also. The rationale for the *r-th moment* $(r = 0,1,2,\ldots)$ of any distribution function can be described as follows. For convenience, we define these moments by adjusting the 0-th moment to unity.

Let $\psi(t)$ be any density function, and $\Psi(t)$ be the corresponding distribution function. We can write

$$(7.3) \qquad \Psi(t) = \int_{-\infty}^{t} \psi(u)\,du \quad .$$

Since $\Psi(-\infty) = 0$ and $\Psi(\infty) = 1$, we must set a finite interval to make the integral of $\Psi(t)$ a finite number. As before, we use $\underline{t}$ $(> -\infty)$ and $\bar{t}$ $(< \infty)$ for the lower and upper endpoints of the interval, respectively. We can write

$$(7.4) \qquad \int_{\underline{t}}^{\bar{t}} \Psi(t)\,dt = t\Psi(t)\Big|_{\underline{t}}^{\bar{t}} - \int_{\underline{t}}^{\bar{t}} t\psi(t)\,dt$$

$$= \bar{t}\Psi(t) - \underline{t}\Psi(t) - \int_{\underline{t}}^{\bar{t}} t\psi(t)\,dt \quad .$$

If we define a new function, $\phi(t)$, such that

$$(7.5) \qquad \phi(t) = \left[\int_{\underline{t}}^{\bar{t}} \Psi(t)\,dt\right]^{-1} \Psi(t) \quad ,$$

this function can be a density function also, with the distribution function, $\Phi(t)$, such that

$$(7.6) \qquad \Phi(t) = \int_{-\infty}^{t} \phi(u)\, du = \int_{\underline{t}}^{t} \phi(u)\, du \quad ,$$

and

$$(7.7) \qquad \begin{cases} \Phi(-\infty) = \Phi(\underline{t}) = 0 \\ \Phi(\infty) = \Phi(\bar{t}) = 1 \quad . \end{cases}$$

It is obvious that the new density function, $\phi(t)$, is strictly increasing in t for the interval, $(\underline{t}, \bar{t})$.

We shall consider the r-th moment of t about any arbitrary, finite number a , when the distribution function is $\Phi(t)$. We define a function, $F_r(\underline{t},\bar{t};a)$, such that

$$(7.8) \qquad F_r(\underline{t},\bar{t};a) = \int_{\underline{t}}^{\bar{t}} (t-a)^r\, \Psi(t)\, dt$$

$$= (r+1)^{-1}(t-a)^{r+1}\, \Psi(t)\Big|_{\underline{t}}^{\bar{t}} - \int_{\underline{t}}^{\bar{t}} (r+1)^{-1}(t-a)^{r+1}\psi(t)\, dt$$

$$= (r+1)^{-1}[(\bar{t}-a)^{r+1}\Psi(\bar{t}) - (\underline{t}-a)^{r+1}\Psi(\underline{t}) - \int_{\underline{t}}^{\bar{t}} (t-a)^{r+1}\psi(t)\, dt] \quad .$$

for $r = 0,1,2,3,\ldots$ It is obvious that $F_0(\underline{t},\bar{t};a)$ equals the term given by (7.4). Thus we can write for the r-th moment about any arbitrary finite number a , which is denoted by $\mu_r^*(\underline{t},\bar{t};a)$,

$$(7.9) \qquad \mu_r^*(\underline{t},\bar{t};a) = F_r(\underline{t},\bar{t};a)[F_0(\underline{t},\bar{t};a)]^{-1} \quad .$$

In a special case where $\Psi(t) \doteq 0$ and $\Psi(t) \doteq 1$, we have

$$(7.10) \qquad F_0(\underline{t},\bar{t};a) \doteq \bar{t} - E(t;\psi) \quad ,$$

where $E(t;\psi)$ indicates the expectation of t when the density function is $\psi(t)$. If, in addition, $(\underline{t}-a)^{r+1}\Psi(\underline{t}) \doteq 0$ and $(\bar{t}-a)^{r+1}\Psi(\bar{t}) \doteq (\bar{t}-a)^{r+1}$, then we can write

$$\mu_r^*(\underline{t},\bar{t};a) = (r+1)^{-1}[(\bar{t}-a)^{r+1} - E\{(t-a)^{r+1};\psi\}] \\ [\bar{t} - E(t;\psi)]^{-1} \quad . \tag{7.11}$$

Table 7-2 presents the two sets of moments, one of which was obtained by the Simpson's Method, and the other through (7.9), naming, for convenience, the former "Empirical" and the latter "Theoretical." In the upper half of the table, some values are marked by *, ** or ***, indicating that the empirical moment is different from the corresponding theoretical moment in absolute value by greater than 1 percent and less than or equal to 2 percent, greater than 2 percent and less than or equal to 3 percent, or greater than 3 percent and less than or equal to 4 percent. None of them have the discrepancies greater than 4 percent. This result shows that the discrepancies are small for the moments of lower degrees, and greater for higher degrees. There also is a tendency that the discrepancies are less for larger intervals of t than for smaller ones, such as [-2.0, 2.0] and [-3.0, 3.0] .

Since we used the same set of empirical moments for both the method of moments and least squares solution procedures, it is unlikely that the rounding errors of the empirical moments affect the results of the least squares solution procedure, to the extent that the resultant curves hardly justify the least squares principle. This is also supported by the fact that the discrepancies are less for the larger intervals of t, the tendency which is opposite to that shown in

the discrepancies between the two sets of resulting coefficients, as is clear in Table 7-1. And yet there is a slight possibility that the rounding errors affect the results of the least squares solution procedure more strongly than those of the method of moments procedure, because of the difference in the mathematical and computational processes involved in these two procedures.

Another possibility to account for the failure of the least squares solution procedure in producing right curves in certain cases is that we used a single precision program to produce the matrix A, although after that step every program was written in double precision. This is hardly considered as a major reason for the failure, but it is worth investigating.

Finally, the third conceivable reason is that we used F-field to punch out the resultant inverse matrix A^{-1}.§ It is very possible that this affected the resultant coefficients for higher powers, and these slight discrepancies in the coefficients for higher powers amplified the discrepancy of the curve by the least squares solution procedure for the values of t further apart from zero.

To investigate these possibilities, we have set the following two cases, in addition to Case 1, and tried them on the least square solution procedure. For Case 3, the theoretical moments, which are shown in Table 7-2, are also used.

Case 1: Matrix A, single precision; A^{-1}, F-field
Case 2: Matrix A, double precision; A^{-1}, F-field
Case 3: Matrix A, double precision; A^{-1}, D-field

§For obtaining A^{-1}, the subprogram, LINV2F of IMSL was used.

TABLE 7-2

First through Seventh Moments about the Midpoints, Which Are Also the Moments about the Origin, Obtained by Simpson's Method (Empirical) and from the Mathematical Formula (Theoretical). Some Empirical Moments Are Marked by *, ** or ***, Depending upon the Discrepancies from the Corresponding Theoretical Moments, i.e., * for 1 through 2 Percent, ** for 2 through 3 Percent, and *** for More Than 3 Percent

Interval		[-2.0, 2.0]	[-3.0, 3.0]	[-4.0, 4.0]	[-6.0, 6.0]	[-8.0, 8.0]	[-10.0, 10.0]
EMPIRICAL	μ_1^*	0.76608	1.32863	1.86877	2.90972	3.93021	4.94250
	μ_2^*	1.32083	2.97778	5.30208	11.95139	21.26771	33.25084
	μ_3^*	1.71238*	6.43702*	15.68153	53.57640	127.37397	249.09253
	μ_4^*	3.13750*	15.97778*	50.66876*	257.40142	814.93447	1991.66771
	μ_5^*	4.69229**	39.09748*	167.91547*	1284.78490	5426.88917	16583.08591
	μ_6^*	8.88036**	102.12066*	576.61171*	6600.34511	37176.08152	142023.82529
	μ_7^*	14.18575***	264.11926**	2010.67977*	34601.01872*	259957.85738	1241665.48597
THEORETICAL	μ_1^*	0.77038	1.33423	1.87502	2.91667	3.93750	4.95000
	μ_2^*	1.33356	3.00003	5.33333	12.00000	21.33333	33.33333
	μ_3^*	1.74127	6.50937	15.81279	53.87500	127.90625	249.92500
	μ_4^*	3.20068	16.20020	51.20001	259.19999	819.19995	1999.99988
	μ_5^*	4.81883	39.76985	170.04675	1295.58303	5461.01953	16666.41269
	μ_6^*	9.14488	104.14413	585.14299	6665.14246	37449.14063	142857.13434
	μ_7^*	14.70112	270.18844	2044.81133	34989.81252	262142.35938	1249998.68750

The two intervals, [-6.0, 6.0] and [-8.0, 8.0], were chosen for the investigation. Table 7-3 presents the coefficients of the polynomials obtained for degrees 3, 4, 5, 6 and 7, in each of the three cases using the empirical moments, as well as the results of Case 2 based on the theoretical moments. We can see that for the degrees, 3 through 5, the resultant coefficients of Cases 1, 2 and 3 are almost identical with those obtained by the method of moments procedure, with the exception of slight discrepancies in Case 1. For the degrees, 6 and 7, the discrepancies are greater, and in Cases 1 and 2 they are substantial. It is obvious from these results that the resultant coefficients of Case 3 are the closest to those of the method of moments procedure, and, in fact, they are practically identical. It is also interesting to note that, in spite of the discrepancies of the coefficients based on the empirical moments from those based on the theoretical moments in the polynomials of lower degrees, such as degrees 3, 4 and 5, for the polynomials of higher degrees the results of Cases 1 and 2 on the empirical moments are close to those of Case 2 based on the theoretical moments, as is well exemplified by -0.00019 for α_7 .

These findings from the resultant coefficients in the various cases indicate that the failure for the least squares solution procedure to produce right curves was due to the handling of the matrix A^{-1} by the F-field, and the use of D-field, which was adopted in Case 3, turned out to be the solution, as long as we use the interval, [-6.0, 6.0]. We have learned that small rounding errors caused in the process of the inversion of the

TABLE 7-3

Coefficients α_i's of the Resultant Polynomials of Degrees 3, 4, 5, 6 and 7 Fitted to the Standard Normal Distribution Function by the Method of Moments (M.M.) and the Least Squares Solution (L.S.), Using the Interval, [-6.0, 6.0]. There Are Four Different Sets of Results for the Least Squares Solution.

		M.M. Empirical	L.S. Empirical Case 1	L.S. Empirical Case 2	L.S. Theoretical Case 2	L.S. Empirical Case 3
DGR. 3	α_0	0.083755	0.083755	0.083755	0.083333	0.083755
	α_1	0.035859	0.035859	0.035859	0.035657	0.035859
	α_2	-0.000035	-0.000035	-0.000035	0.000000	-0.000035
	α_3	-0.000725	-0.000725	-0.000725	-0.000713	-0.000725
DGR. 4	α_0	0.083472	0.083472	0.083472	0.083333	0.083472
	α_1	0.035859	0.035859	0.035859	0.035657	0.035859
	α_2	0.000044	0.000043	0.000044	0.000000	0.000044
	α_3	-0.000725	-0.000725	-0.000725	-0.000713	-0.000725
	α_4	-0.000003	-0.000002	-0.000003	0.000000	-0.000003
DGR. 5	α_0	0.083472	0.083472	0.083472	0.083333	0.083472
	α_1	0.047343	0.047344	0.047343	0.047429	0.047343
	α_2	0.000044	0.000043	0.000044	0.000000	0.000044
	α_3	-0.002213	-0.002214	-0.002213	-0.002239	-0.002213
	α_4	-0.000003	-0.000002	-0.000003	0.000000	-0.000003
	α_5	0.000037	0.000037	0.000037	0.000038	0.000037
DGR. 6	α_0	0.084011	0.084010	0.084012	0.083335	0.084010
	α_1	0.047343	0.047344	0.047343	0.047429	0.047343
	α_2	-0.000271	-0.000273	-0.000271	-0.000001	-0.000270
	α_3	-0.002213	-0.002214	-0.002213	-0.002239	-0.002213
	α_4	0.000024	0.000024	0.000020	-0.000003	0.000024
	α_5	0.000037	0.000037	0.000037	0.000038	0.000037
	α_6	-0.000001	0.000001	0.000001	0.000002	-0.000001
DGR. 7	α_0	0.084011	0.084010	0.084012	0.083335	0.084010
	α_1	0.056338	0.056358	0.056341	0.055619	0.056338
	α_2	-0.000271	-0.000273	-0.000271	-0.000001	-0.000270
	α_3	-0.004462	-0.004462	-0.004446	-0.004269	-0.004462
	α_4	0.000024	0.000024	0.000020	-0.000003	0.000024
	α_5	0.000175	0.000161	0.000171	0.000159	0.000175
	α_6	-0.000001	0.000001	0.000001	0.000002	-0.000001
	α_7	-0.000002	-0.000019	-0.000019	-0.000019	-0.000002

matrix A can make the resultant polynomial far from the right one which follows the least squares principle, a warning which researchers should keep in mind when they use the least squares solution. On the other hand, the method of moments procedure is so straight-forward and uncomplicated, that we need little precaution in using it.

The results of the corresponding investigation for the interval, [-8.0, 8.0] are presented in Table 7-4. It is clear from this table that the use of the theoretical moments, instead of the empirical moments, does not improve the result, nor does the use of the double precision program for producing the matrix A , but the use of the D-field, instead of the F-field, works, as was the case with the interval, [-6.0, 6.0]. We notice, however, that the coefficients of the polynomial of degree 7 are substantially different between the result of the method of moments procedure based on the empirical moments and that of Case 3 based on the empirical moments, in spite of the fact that they are identical for all the other degrees. The resultant polynomials of Case 3 are plotted in Figure 7-4, and, while for the other degrees two curves are not distinguishable, for degree 7 they are clearly two different curves. Note, however, that the polynomial obtained by Case 3 also demonstrates the least squares principle, unlike the one obtained by Case 1 . To pursue this problem further, for the interval, [-8.0, 8.0], two additional processes were taken to obtain the coefficients of the polynomials of degrees 3, 4, 5, 6 and 7 , i.e., the method of moments procedure based on the theo-

TABLE 7-4

Coefficients α_i's of the Resultant Polynomials of Degrees 3, 4, 5, 6 and 7 Fitted to the Standard Normal Distribution Function by the Method of Moments (M.M.) and the Least Squares Solution (L.S.), Using the Interval, [-8.0, 8.0]. There Presented Are Five Different Sets of Results for the Least Squares Solution, and Two Different Sets of Results for the Method of Moments.

		M.M. Theoretical	M.M. Empirical	L.S. Empirical Case 1	L.S. Empirical Case 2	L.S. Theoretical Case 2	L.S. Empirical Case 3	L.S. Theoretical Case 3
DGR. 3	α_0	0.062500	0.062740	0.062740	0.062740	0.062500	0.062740	0.062500
	α_1	0.020866	0.020946	0.020946	0.020946	0.020866	0.020946	0.020866
	α_2	0.0	-0.000011	-0.000011	-0.000011	-0.000000	-0.000011	0.0
	α_3	-0.000243	-0.000246	-0.000246	-0.000246	-0.000243	-0.000246	-0.000243
DGR. 4	α_0	0.062500	0.062590	0.062590	0.062591	0.062500	0.062590	0.062500
	α_1	0.020866	0.020946	0.020946	0.020946	0.020866	0.020946	0.020866
	α_2	-0.000000	-0.000012	0.000012	0.000012	-0.000000	0.000012	-0.000000
	α_3	-0.000243	-0.000246	-0.000246	-0.000246	-0.000243	-0.000246	-0.000243
	α_4	0.0	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	0.0
DGR. 5	α_0	0.062500	0.062590	0.062590	0.062591	0.062500	0.062590	0.062500
	α_1	0.028822	0.028771	0.028772	0.028769	0.028820	0.028771	0.028822
	α_2	-0.000000	0.000012	0.000012	0.000012	-0.000000	0.000012	-0.000000
	α_3	-0.000823	-0.000816	-0.000818	-0.000818	-0.000825	-0.000816	-0.000823
	α_4	0.0	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	0.0
	α_5	0.000008	0.000008	0.000010	0.000010	0.000010	0.000008	0.000008
DGR. 6	α_0	0.062501	0.062912	0.062908	0.062905	0.062493	0.062912	0.062500
	α_1	0.028822	0.028771	0.028772	0.028769	0.028820	0.028771	0.028822
	α_2	-0.000000	-0.000093	-0.000100	-0.000109	-0.000016	-0.000093	-0.000000
	α_3	-0.000823	-0.000816	-0.000818	-0.000818	-0.000825	-0.000816	-0.000823
	α_4	0.0	0.000005	0.000008	0.000008	0.000003	0.000005	0.0
	α_5	0.000008	0.000008	0.000010	0.000010	0.000010	0.000008	0.000008
	α_6	0.0	-0.000000	-0.000017	-0.000017	-0.000017	-0.000000	0.0
DGR. 7	α_0	0.062501	0.062912	0.062908	0.062905	0.062493	0.062912	0.062500
	α_1	0.035278	0.035557	0.035454	0.028669	0.028720	0.028760	0.028812
	α_2	-0.000000	-0.000093	-0.000100	-0.000109	-0.000016	-0.000093	-0.000000
	α_3	-0.001731	-0.001770	-0.001784	-0.000880	-0.000887	-0.000815	-0.000822
	α_4	0.0	0.000005	0.000008	0.000008	0.000003	0.000005	0.0
	α_5	0.000039	0.000041	0.000000	0.000001	0.000001	0.000008	0.000008
	α_6	0.0	-0.000000	-0.000017	-0.000017	-0.000017	-0.000000	0.0
	α_7	-0.000000	-0.000000	-0.000041	-0.000000	-0.000000	0.000000	0.0

retical moments, and Case 3 of the least squares solution procedure based on the theoretical moments. The resultant coefficients are shown in the first and last columns of Table 7-4, and the comparison of these two sets of coefficients clarifies that the same effect is observed in the polynomials of degree 7 . Thus it is clear that the use of D-field does not solve the problem completely for the least squares solution procedure.

VIII More Examples

In the preceding two chapters, the two functions we ha
to fit polynomials to are both of simple nature, i.e., a unim
symmetric function and a strictly increasing function. To p
how the method of moments works as a way of fitting a polynom
to empirical data, further, we shall observe somewhat differe
examples in this chapter. Since the usefulness of the metho
moments goes beyond the realm of psychometrics, we have selec
our data from physics, in which such a curve-fitting method m
a substantial appreciation. In so doing, we have tried to c
semi-artificial data which provide us with empirical functio
more or less complicated nature.

The Maxwell speed density function for a particle in a
derived first by James Clarke Maxwell in 1860. It is stated
without the process of its derivation, for the conditional de
$\psi(v|m,T)$, of the speed v of the particle in meters x se
given the mass m of the molecule in kilograms and the temp
T of the gas in the degrees Kelvin, such that

$$(8.1) \qquad \psi(v|m,T) = 4\pi(m/2\pi kT)^{3/2} v^2 \exp[-mv^2/2kT]$$

where k is Boltzmann's constant, which is experimentally d
to be

$$(8.2) \qquad k = 1.380662 \times 10^{-23} \quad \text{Joules/Kilogram}$$

(e.g., Brown, 1968). If we fix the temparature T , then
density, $\xi(v,m;T)$, of speed and mass is given by

$$\xi(v,m;T) = \psi(v|m,T)\ f(m) \quad , \tag{8.3}$$

where $f(m)$ is the density function of mass. From this, we can write for the marginal density function, $g(v)$, of speed such that

$$\begin{aligned} g(v) &= \int_0^\infty \xi(v,m;T)\ dm \\ &= \int_0^\infty \psi(v|m,T)\ f(m)\ dm \quad . \end{aligned} \tag{8.4}$$

The conditional density function, $\phi(m|v;T)$, of mass, given speed, is obtained, therefore, by

$$\phi(m|v;T) = \xi(v,m;T)\ [g(v)]^{-1} \quad . \tag{8.5}$$

For the purpose described earlier, we "created" our own mixture of twenty-one gas particles, in order to obtain several different pseudo-empirical conditional density functions of mass, given speed. Table 8-1 presents these twenty molecules and the two different probability functions we created. Thus the density function $f(m)$ in (8.3) is replaced by the probability function $p(m)$, (8.4) is replaced by

$$g(v) = \sum_{j=1}^{n} \xi(v,m_j;T) \quad , \tag{8.6}$$

where n is the number of gasses which have positive probabilities, and $\phi(m|v;T)$ becomes a probability function.

In order to avoid large dispersions of these gasses, we shall use the natural logarithm of the atomic mass, M, which is given by

$$M = m\ N_A \quad , \qquad (N_A \doteq 6.022 \times 10^{26}) \tag{8.7}$$

TABLE 8-1

Two "Created" Probability Functions for the Twenty Molecules

Molecule	Mass (Kilograms)	log M	Mixture A	Mixture C
H_2	0.33476E-26	0.6981	0.200	0.300
He	0.66473E-26	1.3863	0.050	0.100
CH_4	0.26640E-25	2.7751	0.020	0.050
Ne	0.33515E-25	3.0047	0.015	0.050
N_2	0.46519E-25	3.3322	0.020	0.100
CO	0.46519E-25	3.3329	0.020	0.022
NO	0.49828E-25	3.4015	0.005	0.020
O_2	0.53136E-25	3.4657	0.010	0.020
F_2	0.63097E-25	3.6376	0.020	0.020
A	0.66337E-25	3.6876	0.150	0.018
N_2O	0.73082E-25	3.7842	0.130	0.010
CO_2	0.73088E-25	3.7846	0.100	0.010
NO_2	0.76396E-25	3.8289	0.030	0.010
O_3	0.78044E-25	3.8501	0.025	0.010
Cl_2	0.11774E-24	4.2613	0.010	0.010
Br	0.13269E-24	4.3809	0.010	0.050
Kr	0.13916E-24	4.4284	0.015	0.100
C_2NO_3	0.14281E-24	4.4543	0.020	0.050
Xe	0.21803E-24	4.8775	0.050	0.025
I_2	0.42147E-24	5.5366	0.100	0.025

E-s indicates the multiplication by 10^{-s} .

where N_A is Avogadro's number, i.e., the number of molecules in a kilogram-molecule of any substance, as our random variable. Since $\phi(m|v;T)$ is the conditional probability function of mass, given speed, it is easy for us to produce complicated ones by adjusting the marginal probability function of mass.

Figures 8-1, 8-2 and 8-3 present the resultant polynomials of degrees 3, 4, 5, 6 and 7 obtained by the method of moments, for the three different $\phi(m|v;T)$'s , which are based on two different mixtures of gasses, A and C . In the same figures, also presented are the conditional probability functions $\phi(m|v;T)$, in the form of histogram with the step width of 0.5 . The fixed speeds for these three examples are 280 meters/second, 480 meters/second, and 1,280 meters/second, respectively. The coefficients of these polynomials shown in Figures 8-1 through 8-3 are presented in Table 8-2. The moments about the midpoint, 3.11008, on which the method of moments were applied, are shown in Table 8-3.

The results of these three examples tell us that we need high degree polynomials to attain a relatively high level of fitness, if the empirical function is of complicated nature. In other words, it is more difficult to approximate bimodal functions like the one in the third example, and trimodal functions like those in the first and second examples, than to approximate unimodal functions like the standard normal density function, and monotone functions like the standard normal distribution function. If, for instance, the two small densities on the left hand side of Figure 8-2 did not exist, we should expect better fitnesses of the polynomials of higher degrees, as Figure 8-1 indicates. This implies another warning

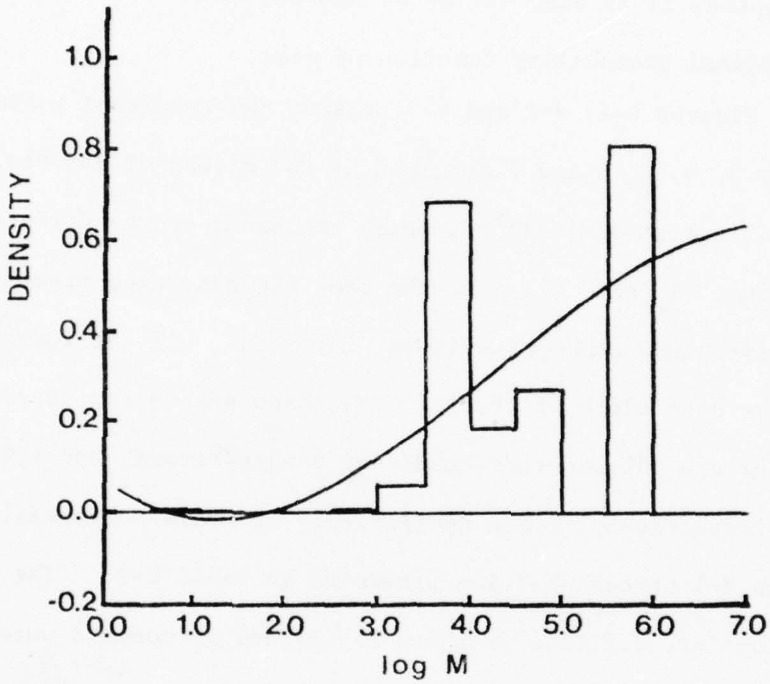

FIGURE 8-1

Polynomials Obtained by the Method of Moments, with the Interval [0.35400, 5.86615], Approximating the Conditional Density of Mass, Given Speed, for Mixture A with 280 Meters/Second As the Fixed Speed. The Polynomial Is of Degree 3.

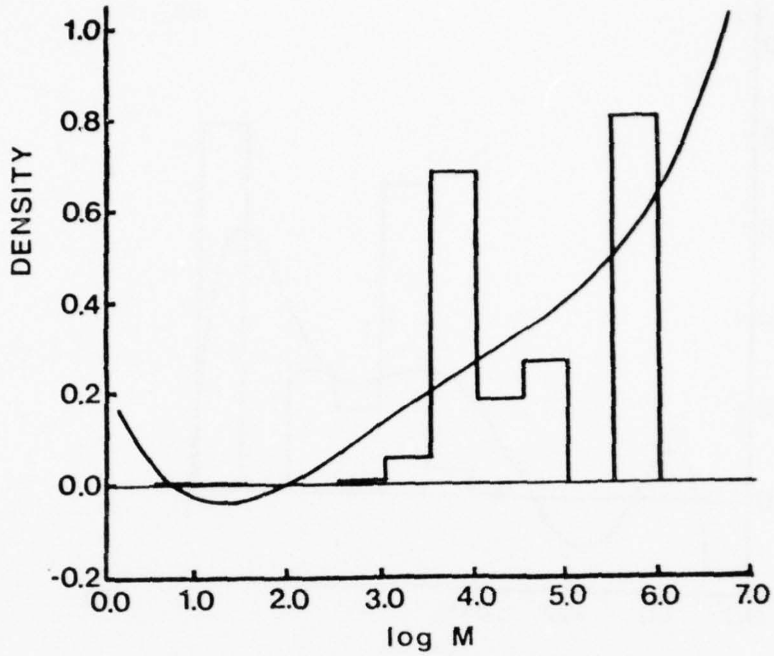

FIGURE 8-1 (Continued): The Polynomial Is of Degree 4.

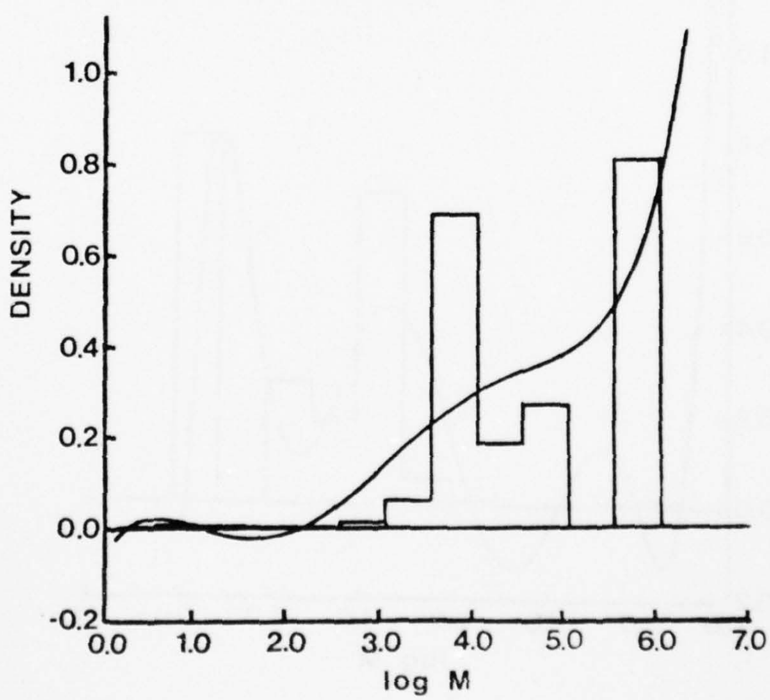

FIGURE 8-1 (Continued): The Polynomial Is of Degree 5.

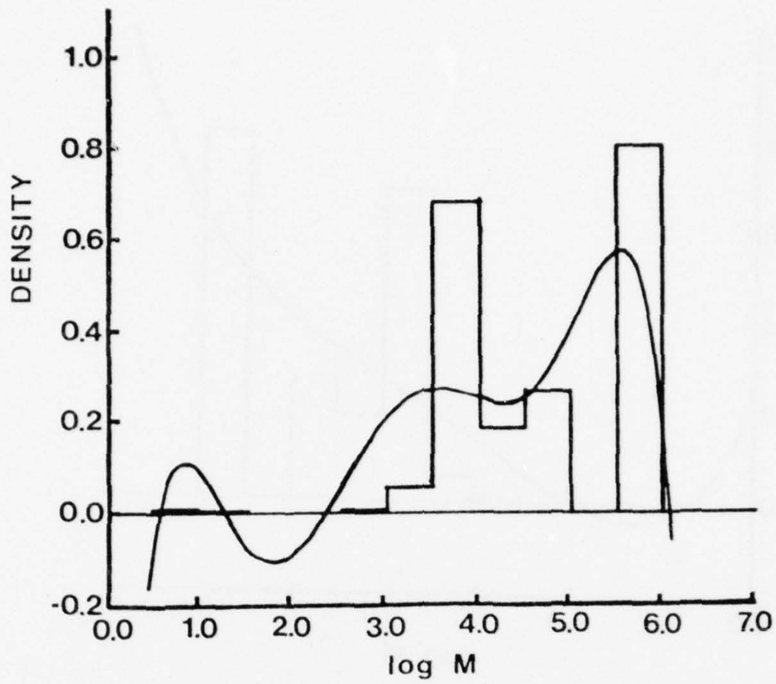

FIGURE 8-1 (Continued): The Polynomial Is of Degree 6.

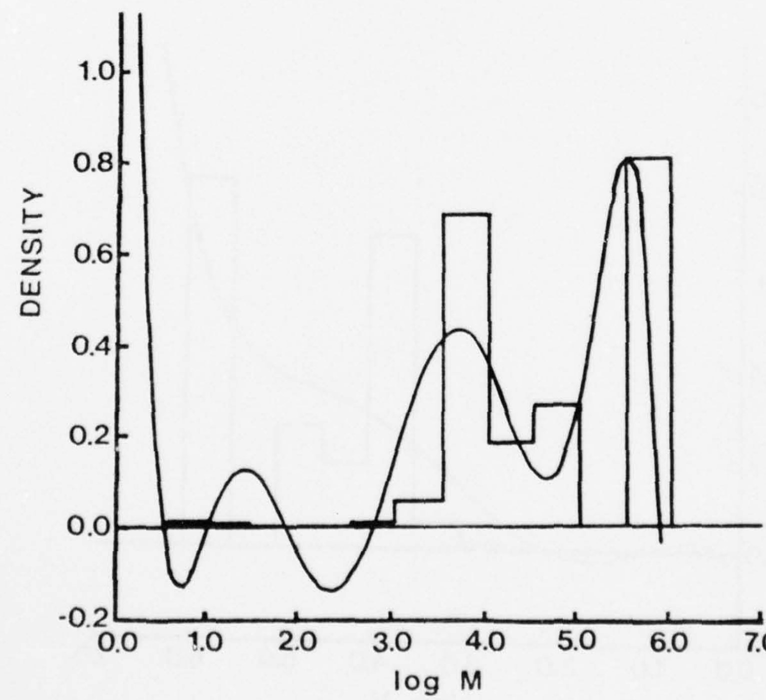

FIGURE 8-1 (Continued): The Polynomial Is of Degree 7.

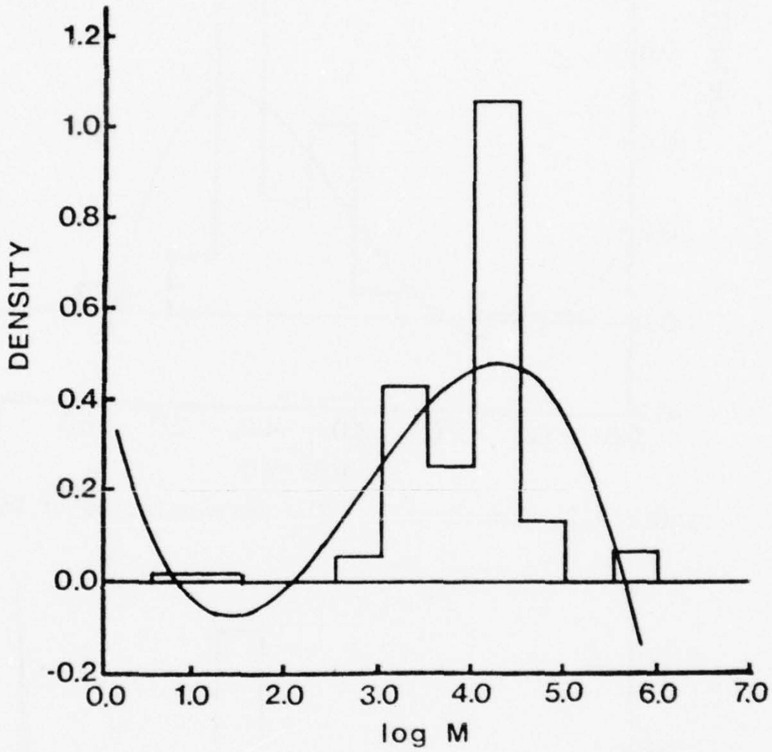

FIGURE 8-2

Polynomials Obtained by the Method of Moments, with the Interval [0.35400, 5.86615], Approximating the Conditional Density of Mass, Given Speed, for Mixture C with 480 Meters/Second As the Fixed Speed. The Polynomial Is of Degree 3.

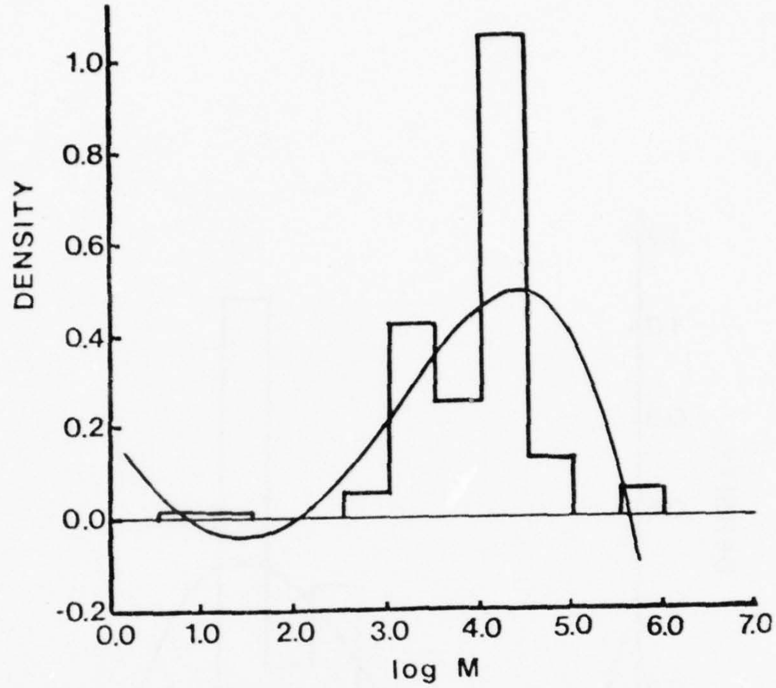

FIGURE 8-2 (Continued): The Polynomial Is of Degree 4.

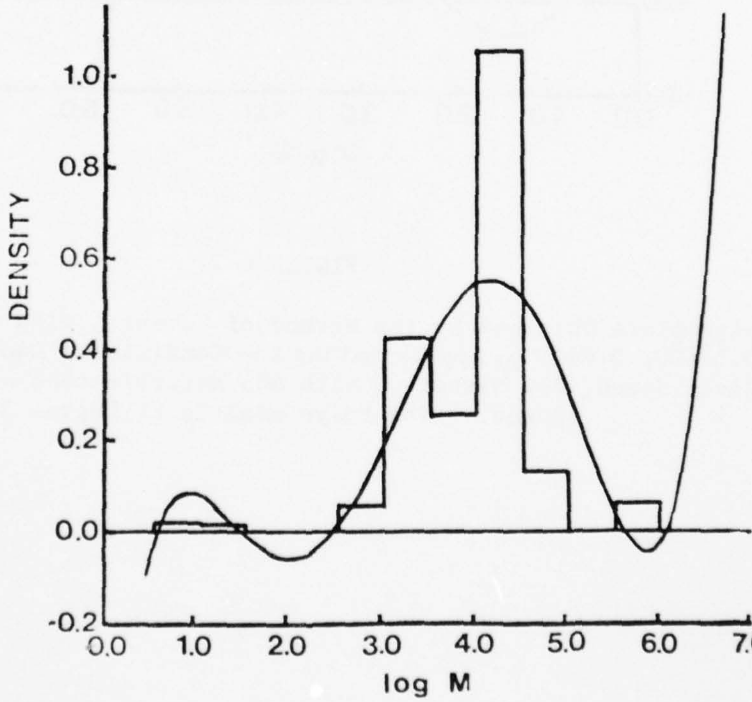

FIGURE 8-2 (Continued). The Polynomial Is of Degree 5.

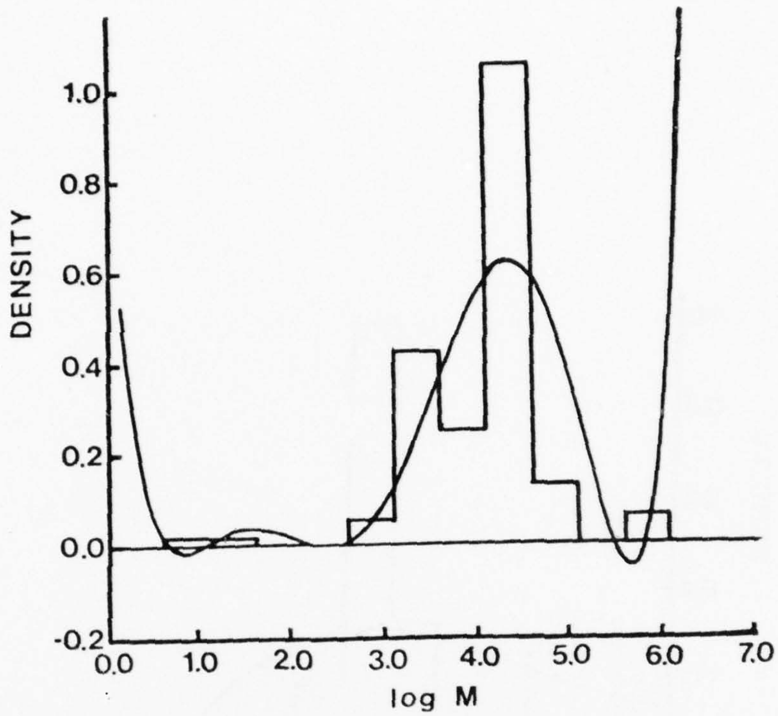

FIGURE 8-2 (Continued): The Polynomial Is of Degree 6.

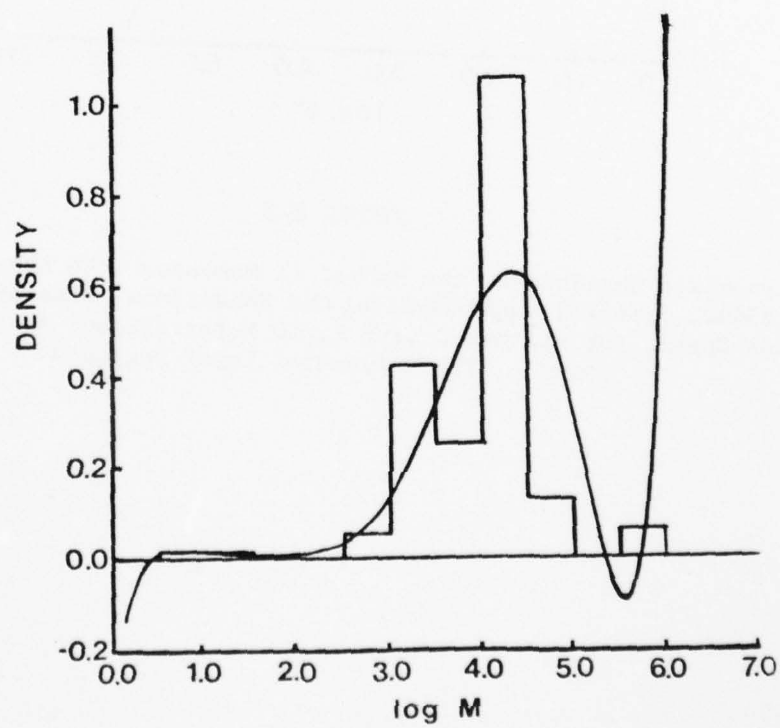

FIGURE 8-2 (Continued): The Polynomial Is of Degree 7.

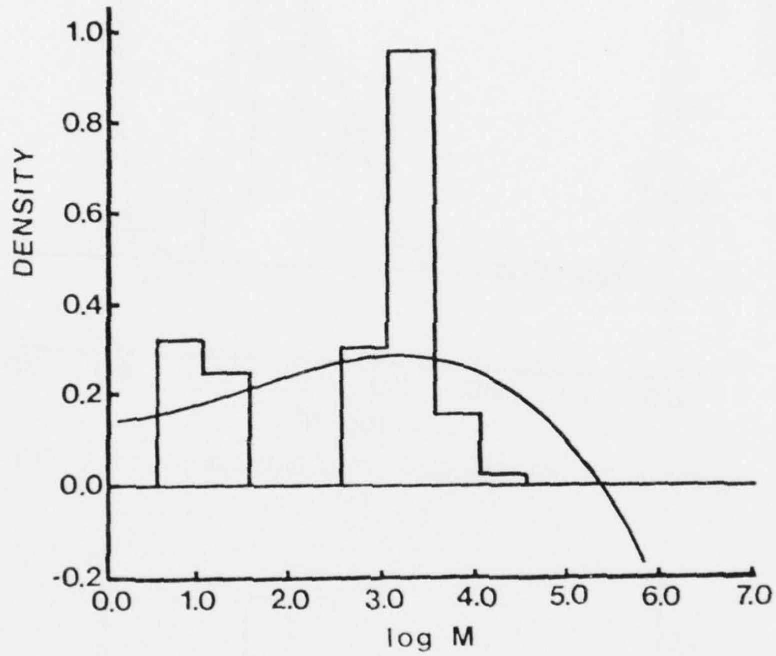

FIGURE 8-3

Polynomials Obtained by the Method of Moments, with the Interval [0.35400, 5.86615], Approximating the Conditional Density of Mass, Given Speed, for Mixture C with 1,280 Meters/Second As the Fixed Speed. The Polynomial Is of Degree 3.

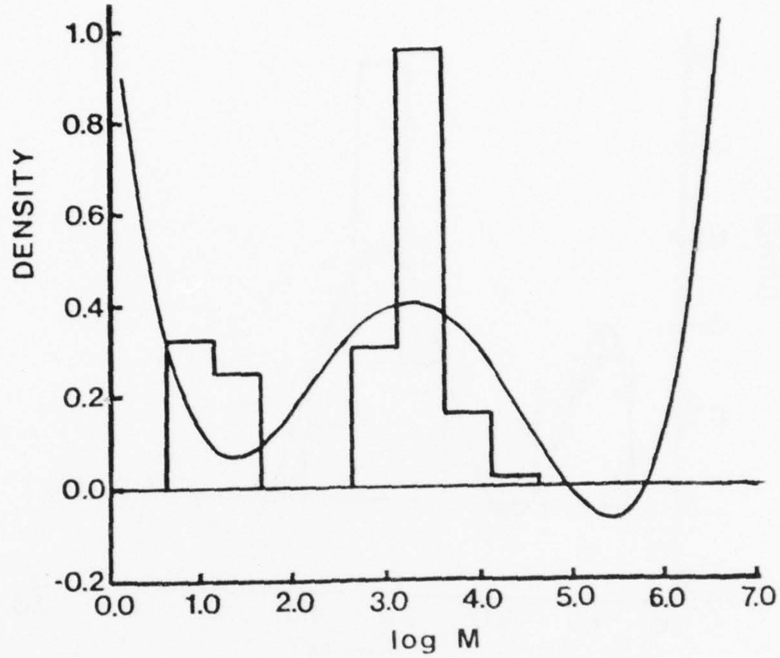

FIGURE 8-3 (Continued): The Polynomial Is of Degree 4.

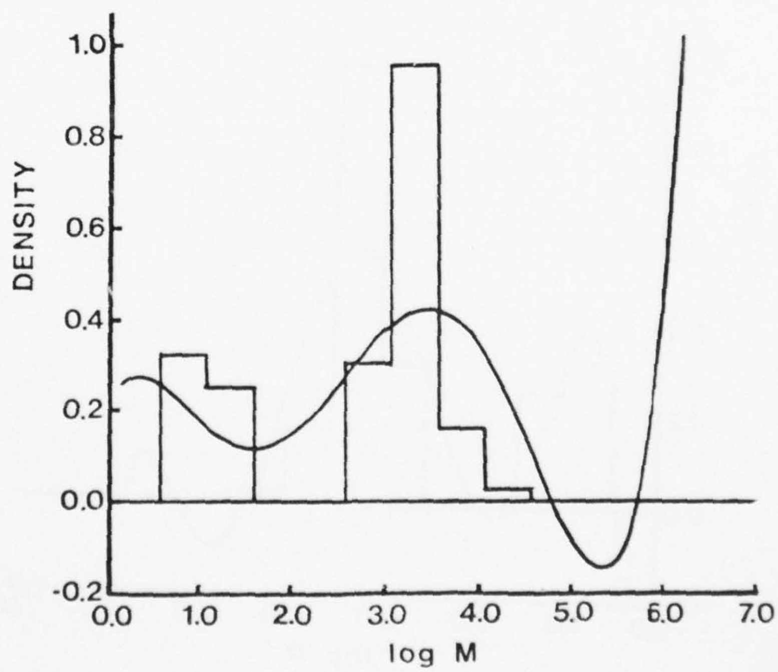

FIGURE 8-3 (Continued): The Polynomial Is of Degree 5.

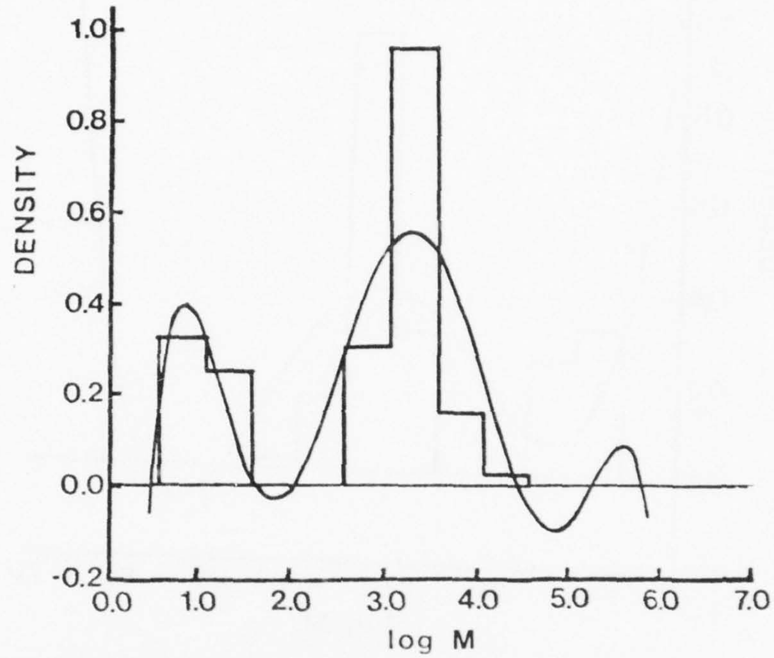

FIGURE 8-3 (Continued): The Polynomial Is of Degree 6.

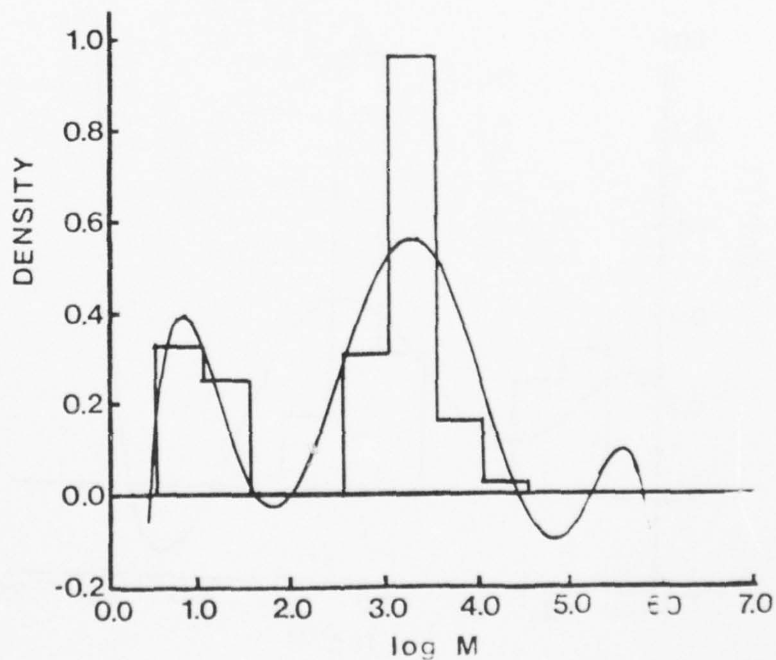

FIGURE 8-3 (Continued): The Polynomial Is of Degree 7.

TABLE 8-2

Coefficients of the Polynomials of Degrees 3, 4, 5, 6 and 7 Fitted to the Three Different Conditional Densities of Log M , Given Speed, by the Method of Moments

Mixture		A	C	C
Speed		280	480	1,280
0	DGR. 3	0.067573	0.404565	0.142486
1		-0.149453	-0.787689	0.008412
2		0.072221	0.380381	0.038139
3		-0.005638	-0.044981	-0.008429
0	DGR. 4	0.211618	0.181325	0.971013
1		-0.473124	-0.286065	-1.853295
2		0.281124	0.056624	1.239719
3		-0.056076	0.033189	-0.298544
4		0.004054	-0.006284	0.023321
0	DGR. 5	-0.060276	-0.757964	0.233470
1		0.333036	2.498913	0.333511
2		-0.475481	-2.557161	-0.812664
3		0.244709	1.072286	0.517371
4		-0.048925	-0.189307	-0.120392
5		0.003407	0.011770	0.009242
0	DGR. 6	-1.547591	0.644076	-2.641408
1		5.806212	-2.660603	10.912960
2		-7.318827	3.894035	-14.040642
3		4.150070	-2.609286	8.066317
4		-1.161832	0.859826	-2.271607
5		0.157547	-0.133538	0.307190
6		-0.008260	0.007787	-0.015967
0	DGR. 7	3.648567	-0.244966	-2.498739
1		-17.005682	1.241760	10.287386
2		28.585113	-2.247603	-13.056410
3		-23.022640	2.038710	7.321534
4		9.835539	-1.021298	-1.970194
5		-2.277150	0.282921	0.240462
6		0.269464	-0.039718	-0.008355
7		-0.012757	0.002182	-0.000350

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METHOD OF MOMENTS AS THE LEAST SQUARES SOLUTION FOR FITTING A P--ETC(U)
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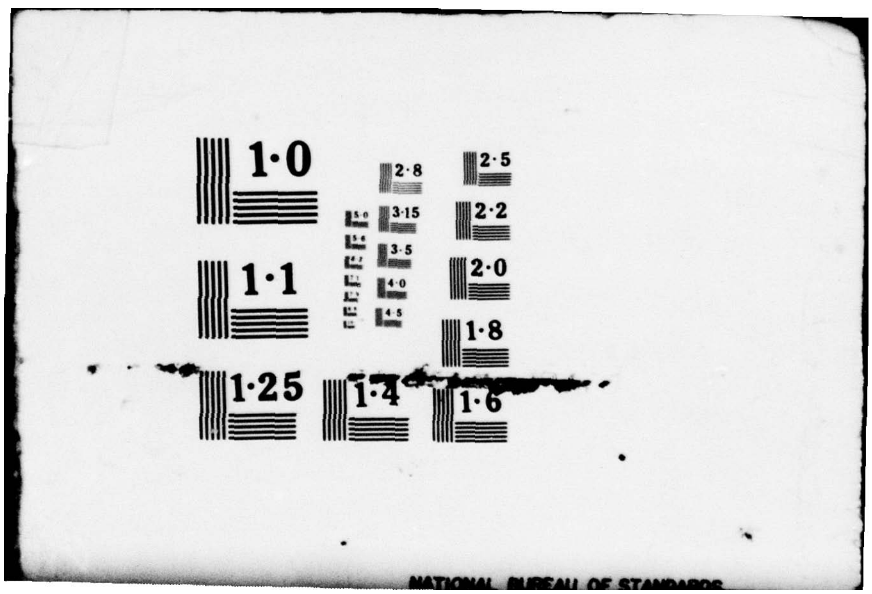
1·0
2·8
2·5
3·15
2·2
3·5
1·1
4·0
2·0
4·5
1·8
1·25
1·4
1·6
NATIONAL BUREAU OF STANDARDS

TABLE 8-3

First through Seventh Moments about the Midpoint 3.11008, for the Three Different Conditonal Densities

Degree of Moments	Mixture A v = 280	Mixture C v = 480	Mixture C v = 1,280
1	1.54209	0.95039	-0.52156
2	3.08669	1.42096	1.38682
3	6.71688	1.84767	-2.84250
4	15.58318	3.63667	6.56796
5	36.20143	4.94297	-14.93710
6	86.71791	12.93124	34.90775
7	205.26370	17.98392	-81.76854

to researchers, i.e., that they should use polynomials of high degrees when the empirical function is multi-modal, or they should divide the interval, for which the function is observed, into an appropriate number of subintervals, and fit a polynomial for each segment of the empirical function, as was done in two earlier examples in Chapter .

IX Discussion and Conclusion

In the present paper, the relationship between the method of moments for fitting a polynomial to a given function and the least squares principle has been observed, and it has been shown that such a polynomial equals the one obtained by the least squares solution. This fact has expanded the meaning of the method of moments, and it is no longer just a method of fitting a specific function to a frequency distribution, or to a set of observations, but a useful way of fitting a polynomial to any empirically obtained function, which is often given in a non-mathematical form. To give an example, when we have succeeded in obtaining the estimated operating characteristics of the item response categories without assuming any mathematical forms, the resultant operating characteristics have no mathematical forms. To analyze them, or to use them for further research, it may be necessary to approximate them by some general mathematical functions. For such purposes, polynomials may be best because of their simple and differentiable nature.

The comparison with the direct least squares solution has proved that the method of moments has an advantage of producing more accurate results, since it does not include the inversion of a matrix, and so forth. Also more practical considerations with respect to the selection of an optimal interval in applying the method of moments, as well as the direct least squares solution, have been made, and it has been clarified that its influence can be devastating. This is also the case in the computation of the moments themselves, especially

of those of higher degrees. Some warnings have been made with respect to the balancing of the sample size and the degree of the polynomial to be searched.

The present findings directly lead us to the prospect of using polynomials in the Conditional P.D.F. Approach and the Bivariate P.D.F. Approach as the estimated conditional density of latent trait θ, given $\hat{\theta}$, instead of Pearson Type density functions, in the estimation of the operating characteristics of the item response categories. If this proves to be successful, then we will be able to establish another method, which is much simpler than Pearson System Method, and yet likely to be more accurate.

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APPENDIX I

Relationship between the Moments about the Midpoint and the Coefficients of the Polynomial Obtained by the Method of Moments

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(1) Transformation of the Variable t to the Variable t^*

Let us transform the variable t into another variable t^* such that

$$(1.1) \qquad t^* = t - M(t) ,$$

where $M(t)$ is the midpoint of t, and is given by

$$(1.2) \qquad M(t) = (\underline{t} + \bar{t}) / 2 ,$$

and $\underline{t}$ and $\bar{t}$ are the greatest lower bound and the least upper bounds of the interval of t. It is obvious that we can write

$$(1.3) \qquad M(t^*) = 0 ,$$

$$(1.4) \qquad \underline{t}^* = - R(t)$$

and

$$(1.5) \qquad \bar{t}^* = R(t) ,$$

where $R(t)$ is one-half of the length of the interval of t. We notice that the m-th moment of t about the midpoint $M(t)$ equals the m-th moment of t^* about the origin. For simplicity, hereafter, we shall use R for $R(t)$.

(2) Relationship between the Moments of t about the Midpoint and the Coefficients of the Polynomial

It is obvious that any polynomial of t with degree i can be expressed as a polynomial of t^* with the same degree. Thus we can write

$$(2.1) \qquad \sum_{i=0}^{m} \alpha_i t^i = \sum_{i=0}^{m} a_i t^{*i} .$$

Let μ^*_{2g} be an even moment about the midpoint of t, for

$$(2.2) \qquad g = 1, 2, \ldots\ldots, [m/2] ,$$

where [ ] denotes the integer part of the number. We can write for this even moment

$$(2.3) \qquad \mu^*_{2g} = \int_{\underline{t}}^{\bar{t}} \{t - M(t)\}^{2g} \sum_{i=0}^{m} \alpha_i t^i \, dt$$

$$= \int_{-R}^{R} t^{*2g} \sum_{i=0}^{m} a_i t^{*i} \cdot \frac{dt}{dt^*} dt^*$$

$$= \sum_{i=0}^{m} a_i \int_{-R}^{R} t^{*2g+i} dt^* .$$

Since we have

$$(2.4) \qquad \int_{-R}^{R} t^{*2g+i} dt^* = (2g+i+1)^{-1} \cdot t^{*2g+i+1} \Big|_{-R}^{R}$$

$$= (2g+i+1)^{-1} [R^{2g+i+1} - (-R)^{2g+i+1}]$$

$$\begin{cases} = 0 & \text{if } i \text{ is odd} \\ = 2(2g+i+1)^{-1} R^{2g+i+1} & \text{if } i \text{ is even ,} \end{cases}$$

we obtain

$$(2.5) \qquad \mu^*_{2g} = 2 \sum_{k=0}^{[m/2]} a_{2k} (2g+2k+1)^{-1} R^{2g+2k+1} .$$

For an odd moment μ^*_{2g+1}, replacing $2g$ in (2.3) and (2.4) by $2g+1$ and switch "odd" and "even" in the last expression of (2.5), we can write

(2.6) $$\mu^*_{2g+1} = 2 \sum_{k=0}^{[(m-1)/2]} a_{2k+1} (2g+2k+3)^{-1} R^{2g+2k+3} .$$

(3) <u>Some Examples of the Relationship between the Set of Moments of t about the Midpoint and the Coefficients a_i's Observed for Polynomials of Different Degrees</u>

It is obvious from (2.4) and (2.5) that the relationship between the set of moments of t about the midpoint and the coefficients a_i's depends upon m. The following are some examples for the polynomials of degrees 3 through 7, which are frequently used in practice.

<u>Degree</u> 3:

(3.1) $$\begin{cases} \mu^*_0 = 2\{Ra_0 + (R^3/3)a_2\} \\ \mu^*_2 = 2\{(R^3/3)a_0 + (R^5/5)a_2\} \end{cases}$$

<u>Degrees</u> 3 <u>and</u> 4:

(3.2) $$\begin{cases} \mu^*_1 = 2\{(R^3/3)a_1 + (R^5/5)a_3\} \\ \mu^*_3 = 2\{(R^5/5)a_1 + (R^7/7)a_3\} \end{cases}$$

<u>Degrees</u> 4 <u>and</u> 5:

(3.3) $$\begin{cases} \mu^*_0 = 2\{Ra_0 + (R^3/3)a_2 + (R^5/5)a_4\} \\ \mu^*_2 = 2\{(R^3/3)a_0 + (R^5/5)a_2 + (R^7/7)a_4\} \\ \mu^*_4 = 2\{(R^5/5)a_0 + (R^7/7)a_2 + (R^9/9)a_4\} \end{cases}$$

<u>Degrees</u> 5 <u>and</u> 6:

(3.4) $$\begin{cases} \mu^*_1 = 2\{(R^3/3)a_1 + (R^5/5)a_3 + (R^7/7)a_5\} \\ \mu^*_3 = 2\{(R^5/5)a_1 + (R^7/7)a_3 + (R^9/9)a_5\} \\ \mu^*_5 = 2\{(R^7/7)a_1 + (R^9/9)a_3 + (R^{11}/11)a_5\} \end{cases}$$

Degrees 6 and 7:

$$
(3.5)\quad \begin{cases}
\mu_0^* = 2\{Ra_0 + (R^3/3)a_2 + (R^5/5)a_4 + (R^7/7)a_6\} \\
\mu_2^* = 2\{(R^3/3)a_0 + (R^5/5)a_2 + (R^7/7)a_4 + (R^9/9)a_6\} \\
\mu_4^* = 2\{(R^5/5)a_0 + (R^7/7)a_2 + (R^9/9)a_4 + (R^{11}/11)a_6\} \\
\mu_6^* = 2\{(R^7/7)a_0 + (R^9/9)a_2 + (R^{11}/11)a_4 + (R^{13}/13)a_6\}
\end{cases}
$$

Degree 7:

$$
(3.6)\quad \begin{cases}
\mu_1^* = 2\{(R^3/3)a_1 + (R^5/5)a_3 + (R^7/7)a_5 + (R^9/9)a_7\} \\
\mu_3^* = 2\{(R^5/5)a_1 + (R^7/7)a_3 + (R^9/9)a_5 + (R^{11}/11)a_7\} \\
\mu_5^* = 2\{(R^7/7)a_1 + (R^9/9)a_3 + (R^{11}/11)a_5 + (R^{13}/13)a_7\} \\
\mu_7^* = 2\{(R^9/9)a_1 + (R^{11}/11)a_3 + (R^{13}/13)a_5 + (R^{15}/15)a_7\}
\end{cases}
$$

APPENDIX II

The Process of Obtained the Matrices $(1/2)\ C_{(0)}^{-1}$ and $(1/2)\ C_{(1)}^{-1}$ for the Polynomials of Degrees 6 and 7 Approximating Any Given Function in the Method of Moments

RESULTS

POLYNOMIAL OF DEGREE 6

(EVEN)

DENOMINATORS OF ORIGINAL MATRIX

1	3	5	7
3	5	7	9
5	7	9	11
7	9	11	13

DENOMINATORS OF ADJOINT MATRIX

280945665	93648555	42567525	22920975
93648555	17342325	6621615	3274425
42567525	6621615	2321865	1091475
22920975	3274425	1091475	496125

NUMERATORS OF ADJOINT MATRIX

256	-768	768	-256
-768	2304	-2304	768
768	-2304	2304	-768
-256	768	-768	256

(ODD)

DENOMINATORS OF ORIGINAL MATRIX

3	5	7
5	7	9
7	9	11

DENOMINATORS OF ADJOINT MATRIX

6237	3465	2205
3465	1617	945
2205	945	525

NUMERATORS OF ADJOINT MATRIX

4	-8	4
-8	16	-8
4	-8	4

(EVEN)

ADJOINT MATRIX (*10**5)

0.09112082	-0.82008740	1.80419228	-1.11688094
-0.82008740	13.28541588	-34.79513684	23.45449964
1.80419228	-34.79513684	99.23057542	-70.36349893
-1.11688094	23.45449964	-70.36349893	51.59989922

DETERMINANT (*10**5) = 0.019042392

INVERSE

4.7851562	-43.0664062	94.7460937	-58.6523437
-43.0664062	697.6757812	-1827.2460937	1231.6992187
94.7460937	-1827.2460937	5211.0351562	-3695.0976562
-58.6523437	1231.6992187	-3695.0976562	2709.7382812

COEFFICIENT MATRIX

2.3925781	-21.5332031	47.3730469	-29.3261719
-21.5332031	348.8378906	-913.6230469	615.8496094
47.3730469	-913.6230469	2605.5175781	-1847.5488281
-29.3261719	615.8496094	-1847.5488281	1354.8691406

(ODD)

ADJOINT MATRIX (*10**5)

64.13339747	-230.88023088	181.40589569
-230.88023088	989.48670377	-846.56084656
181.40589569	-846.56084656	761.90476190

DETERMINANT (*10**5) = 1.116880935

INVERSE

57.4218750	-206.7187500	162.4218750
-206.7187500	885.9375000	-757.9687500
162.4218750	-757.9687500	682.1718750

COEFFICIENT MATRIX

28.7109375	-103.3593750	81.2109375
-103.3593750	442.9687500	-378.9843750
81.2109375	-378.9843750	341.0859375

CHECKING PROCESS

(EVEN)

PRODUCT OF ORIGINAL MATRIX AND ITS ADJOINT (*10**5)

0.01904239	-0.00000000	0.00000000	-0.00000000
-0.00000000	0.01904239	0.0	-0.00000000
0.0	-0.00000000	0.01904239	-0.00000000
-0.00000000	-0.00000000	0.0	0.01904239

(ODD)

PRODUCT OF ORIGINAL MATRIX AND ITS ADJOINT (*10**5)

1.11688094	0.00000000	-0.00000000
0.00000000	1.11688094	-0.00000000
0.0	0.00000000	1.11688094

RESULTS

POLYNOMIAL OF DEGREE 7

(EVEN)

DENOMINATORS OF ORIGINAL MATRIX

1	3	5	7
3	5	7	9
5	7	9	11
7	9	11	13

DENOMINATORS OF ADJOINT MATRIX

280945665	93648555	42567525	22920975
93648555	17342325	6621615	3274425
42567525	6621615	2321865	1091475
22920975	3274425	1091475	496125

NUMERATORS OF ADJOINT MATRIX

256	-768	768	-256
-768	2304	-2304	768
768	-2304	2304	-768
-256	768	-768	256

(ODD)

DENOMINATORS OF ORIGINAL MATRIX

3	5	7	9
5	7	9	11
7	9	11	13
9	11	13	15

DENOMINATORS OF ADJOINT MATRIX

1913106195	869593725	468242775	280945665
869593725	332026695	164189025	93648555
468242775	164189025	77182875	42567525
280945665	93648555	42567525	22920975

NUMERATORS OF ADJOINT MATRIX

256	-768	768	-256
-768	2304	-2304	768
768	-2304	2304	-768
-256	768	-768	256

(EVEN)

ADJOINT MATRIX (*10**5)

0.09112082	-0.82008740	1.80419228	-1.11688094
-0.82008740	13.28541588	-34.79513684	23.45449964
1.80419228	-34.79513684	99.23057542	-70.36349893
-1.11688094	23.45449964	-70.36349893	51.59989922

DETERMINANT (*10**5) = 0.019042392

INVERSE

4.7851562	-43.0664062	94.7460937	-58.6523437
-43.0664062	697.6757812	-1827.2460937	1231.6992187
94.7460937	-1827.2460937	5211.0351562	-3695.0976562
-58.6523437	1231.6992187	-3695.0976562	2709.7382812

COEFFICIENT MATRIX

2.3925781	-21.5332031	47.3730469	-29.3261719
-21.5332031	348.8378906	-913.6230469	615.8496094
47.3730469	-913.6230469	2605.5175781	-1847.5488281
-29.3261719	615.8496094	-1847.5488281	1354.8691406

(ODD)

ADJOINT MATRIX (*10**5)

0.01338138	-0.08831710	0.16401748	-0.09112082
-0.08831710	0.69392011	-1.40326066	0.82008740
0.16401748	-1.40326066	2.98511814	-1.80419228
-0.09112082	0.82008740	-1.80419228	1.11688094

DETERMINANT (*10**5) = 0.000103572

INVERSE

129.1992188	-852.7148438	1583.6132813	-879.7851563
-852.7148438	6699.9023438	-13548.6914063	7918.0664063
1583.6132813	-13548.6914063	28821.7617188	-17419.7460938
-879.7851563	7918.0664063	-17419.7460938	10783.6523438

COEFFICIENT MATRIX

64.5996094	-426.3574219	791.8066406	-439.8925781
-426.3574219	3349.9511719	-6774.3457031	3959.0332031
791.8066406	-6774.3457031	14410.8808594	-8709.8730469
-439.8925781	3959.0332031	-8709.8730469	5391.8261719

CHECKING PROCESS

(EVEN)

PRODUCT OF ORIGINAL MATRIX AND ITS ADJOINT (*10**5)

0.01904239	-0.00000000	0.00000000	-0.00000000
-0.00000000	0.01904239	0.0	-0.00000000
0.0	-0.00000000	0.01904239	-0.00000000
-0.00000000	-0.00000000	0.0	0.01904239

(ODD)

PRODUCT OF ORIGINAL MATRIX AND ITS ADJOINT (*10**5)

0.00010357	-0.00000000	0.0	0.00000000
0.00000000	0.00010357	0.00000000	0.0
0.00000000	0.0	0.00010357	0.00000000
0.00000000	0.00000000	0.0	0.00010357

APPENDIX III

Relationships between Latent Trait θ and Its Estimate λ When λ Satisfies Certain Conditions

I Fundamental Concepts and Formulae

θ : latent trait $\quad -\infty < \theta < \infty$

λ : estimate of θ

η : error of estimation

$f(\theta)$: density function of θ

$\psi(\lambda|\theta)$: conditional density of λ , given θ

$\xi(\lambda,\theta)$: bivariate density of λ and θ

$g(\lambda)$: density function of λ

$\phi(\theta|\lambda)$: conditional density of θ , given λ

(1.1) $\quad \lambda = \theta + \eta$

(1.2) $\quad \xi(\lambda,\theta) = \psi(\lambda|\theta)f(\theta) = \phi(\theta|\lambda)g(\lambda)$

(1.3) $\quad g(\lambda) = \int_{-\infty}^{\infty} \xi(\lambda,\theta)\, d\theta$

II The Relationship between the Conditional Moments of θ and the Conditional Moments of η , Given λ

(1) Moments about the Origin

(2.1.1) $\quad E(\theta|\lambda) = \lambda - E(\eta|\lambda)$

(2.1.2) $\quad E(\theta^2|\lambda) = \lambda^2 - 2\lambda E(\eta|\lambda) + E(\eta^2|\lambda)$

(2.1.3) $\quad E(\theta^3|\lambda) = \lambda^3 - 3\lambda^2 E(\eta|\lambda) + 3\lambda E(\eta^2|\lambda) - E(\eta^3|\lambda)$

(2.1.4) $\quad E(\theta^4|\lambda) = \lambda^4 - 4\lambda^3 E(\eta|\lambda) + 6\lambda^2 E(\eta^2|\lambda)$

$$- 4\lambda E(\eta^3|\lambda) + E(\eta^4|\lambda)$$

(2) Moments about the Mean

(2.2.1) $\mu_n(\theta|\lambda) = (-1)^n \mu_n(\eta|\lambda)$

n-th conditional moment about the mean

In particular:

(2.2.2) $\mu_2(\theta|\lambda) = \mu_2(\eta|\lambda)$

(2.2.3) $\mu_3(\theta|\lambda) = -\mu_3(\eta|\lambda)$

(2.2.4) $\mu_4(\theta|\lambda) = \mu_4(\eta|\lambda)$

III Derivatives of $\log g(\lambda)$

(3.1) $$\frac{d}{d\lambda} \log g(\lambda) = g'/g$$

(3.2) $$\frac{d^2}{d\lambda^2} \log g(\lambda) = [g''(\lambda)]\,[g(\lambda)]^{-1} - [\frac{d}{d\lambda} \log g(\lambda)]^2$$

$$= [g''/g] - [g'/g]^2$$

(3.3) $$\frac{d^3}{d\lambda^3} \log g(\lambda) = [g'''(\lambda)]\,[g(\lambda)]^{-1}$$

$$- 3[\frac{d^2}{d\lambda^2} \log g(\lambda)][\frac{d}{d\lambda} \log g(\lambda)]$$

$$- [\frac{d}{d\lambda} \log g(\lambda)]^3$$

$$= [g'''/g] - 3[g''/g]\,[g'/g] + 2[g'/g]^3$$

(3.4) $$\frac{d^4}{d\lambda^4} \log g(\lambda) = [g''''(\lambda)]\,[g(\lambda)]^{-1}$$

$$- 4[\frac{d^3}{d\lambda^3} \log g(\lambda)][\frac{d}{d\lambda} \log g(\lambda)]$$

$$- 3[\frac{d^2}{d\lambda^2} \log g(\lambda)]^2$$

$$- 6[\frac{d^2}{d\lambda^2} \log g(\lambda)][\frac{d}{d\lambda} \log g(\lambda)]^2$$

$$- [\frac{d}{d\lambda} \log g(\lambda)]^4$$

$$= [g''''/g] - 4[g'''/g][g'/g]$$
$$- 3[g'' \quad]^2 + 12[g''/g]\,[g'/g]^2 - 6[g'/g]^4$$

IV <u>When</u> $\psi(\eta|\theta)$ <u>Is</u> $\eta(0,\sigma)$

(1) <u>Conditional Moments of</u> η <u>about the Origin</u>

(4.1.1) $E(\eta|\lambda) = -\sigma^2[\frac{d}{d\lambda} g(\lambda)][g(\lambda)]^{-1}$

(4.1.2) $E(\eta^2|\lambda) = \sigma^4[\frac{d^2}{d\lambda^2} g(\lambda)][g(\lambda)]^{-1} + \sigma^2$

(4.1.3) $E(\eta^3|\lambda) = 3\sigma^2 E(\eta|\lambda) - \sigma^6[\frac{d^3}{d\lambda^3} g(\lambda)][g(\lambda)]^{-1}$

(4.1.4) $E(\eta^4|\lambda) = 6\sigma^2 E(\eta^2|\lambda) + \sigma^8[\frac{d^4}{d\lambda^4} g(\lambda)][g(\lambda)]^{-1} - 3\sigma^4$

(2) <u>Conditional Moments of</u> θ <u>about the Origin</u>

(4.2.1) $E(\theta|\lambda) = \lambda + \sigma^2[\frac{d}{d\lambda} g(\lambda)][g(\lambda)]^{-1}$

(4.2.2) $E(\theta^2|\lambda) = \lambda^2 + \sigma^2 + 2\lambda\sigma^2[\frac{d}{d\lambda} g(\lambda)][g(\lambda)]^{-1}$
$\qquad + \sigma^4[\frac{d^2}{d\lambda^2} g(\lambda)][g(\lambda)]^{-1}$

(4.2.3) $E(\theta^3|\lambda) = \lambda^3 + 3\lambda\sigma^2 + 3\sigma^2(\lambda^2+\sigma^2)[\frac{d}{d\lambda} g(\lambda)][g(\lambda)]^{-1}$
$\qquad + 3\lambda\sigma^4[\frac{d^2}{d\lambda^2} g(\lambda)][g(\lambda)]^{-1}$
$\qquad + \sigma^6[\frac{d^3}{d\lambda^3} g(\lambda)][g(\lambda)]^{-1}$

(4.2.4) $E(\theta^4|\lambda) = \lambda^4 + 6\lambda^2\sigma^2 + 3\sigma^4$
$\qquad + 4\lambda\sigma^2(\lambda^2+3\sigma^2)[\frac{d}{d\lambda} g(\lambda)][g(\lambda)]^{-1}$
$\qquad + 6\sigma^4(\lambda^2+\sigma^2)[\frac{d^2}{d\lambda^2} g(\lambda)][g(\lambda)]^{-1}$
$\qquad + 4\lambda\sigma^6[\frac{d^3}{d\lambda^3} g(\lambda)][g(\lambda)]^{-1}$
$\qquad + \sigma^8[\frac{d^4}{d\lambda^4} g(\lambda)][g(\lambda)]^{-1}$

(3) Conditional Moments of θ about the Mean

$$(4.3.1)\quad \mu_2(\theta|\lambda) = \sigma^2[1 + \sigma^2\{\frac{d^2}{d\lambda^2}\log g(\lambda)\}]$$
$$= \sigma^2[\sigma^2(g''/g) - \sigma^2(g'/g)^2 + 1]$$

$$(4.3.2)\quad \mu_3(\theta|\lambda) = \sigma^6[\frac{d^3}{d\lambda^3}\log g(\lambda)]$$
$$= \sigma^6[(g'''/g) - 3(g''/g)(g'/g) + 2(g'/g)^3]$$

$$(4.3.3)\quad \mu_4(\theta|\lambda) = \sigma^4[3 + 6\sigma^2\{\frac{d^2}{d\lambda^2}\log g(\lambda)\}$$
$$+ 3\sigma^4\{\frac{d^2}{d\lambda^2}\log g(\lambda)\}^2$$
$$+ \sigma^4\{\frac{d^4}{d\lambda^4}\log g(\lambda)\}]$$
$$= \sigma^4[\sigma^4(g''''/g) - 4\sigma^4(g'''/g)(g'/g)$$
$$+ 6\sigma^4(g''/g)(g'/g)^2$$
$$+ 6\sigma^2(g''/g) - 3\sigma^4(g'/g)^4$$
$$- 6\sigma^2(g'/g)^2 + 3]$$

V General Relationships between the Moments about Any Parameter and Those about the Origin

μ'_n : n-th moment about the origin

μ^*_n : n-th moment about any parameter a

$$(5.1)\quad \mu^*_n = \sum_{r=0}^{n} (-1)^{n-r}\binom{n}{r} a^{n-r}\mu'_r$$

In particular:

$$(5.2)\quad \mu^*_1 = \mu'_1 - a$$

$$(5.3)\quad \mu^*_2 = \mu'_2 - 2a\mu'_1 + a^2$$

$$(5.4)\quad \mu^*_3 = \mu'_3 - 3a\mu'_2 + 3a^2\mu'_1 - a^3$$

(5.5) $\mu_4^* = \mu_4' - 4a\mu_3' + 6a^2\mu_2' - 4a^3\mu_1' + a^4$

(5.6) $\mu_5^* = \mu_5' - 5a\mu_4' + 10a^2\mu_3' - 10a^3\mu_2' + 5a^4\mu_1' - a^5$

In particular, let μ_n denote the n-th moment about the mean.

(5.7) $\mu_1 = 0$

(5.8) $\mu_2 = \mu_2' - \mu_1'^2$

(5.9) $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

(5.10) $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

(5.11) $\mu_5 = \mu_5' - 5\mu_4'\mu_1' + 10\mu_3'\mu_1'^2 - 10\mu_2'\mu_1'^3 + 4\mu_1'^5$

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